ELIMINATION OF SKIDDING DUE TO STEERING MECHANISM ON MOTOR CARS

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When a wheel rolls along on the ground, if there is motion in the direction perpendicular to the plane of the wheel, then there is some sliding on the ground as well as a rolling forward in the plane of the wheel. This sliding is what I shall call "skidding." When a wheel rolls along a curved path, if the plane of the wheel is always tangent to its path, there is no skidding, but only a slight rotation about the point of contact with the ground, for the center of curvature of the path will then be on the projection of the axis of the wheel upon the ground.

In the ordinary wagon, or traction engine, the front axle turns on a single pivot at the center, so that the axes of the front wheels are always in the same straight line, viz., that of the front axle. If such a vehicle is set in motion, and the angle θ (Fig. 1) between the axles is kept constant, the four wheels, A, B, C, D, will describe concentric circles with center at O, the intersection of the lines of the axles, produced. Under these conditions there will be no skidding.

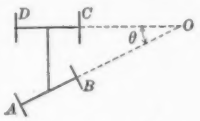


Fig. 1.

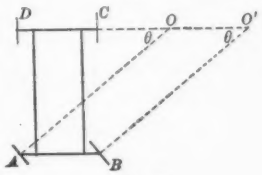


Fig. 2.

But in the automobile each front wheel turns on a separate pivot at the ends of the front axle. Hence the axes of the two front wheels are never in the same line, except when the car is moving straight forward. Let the two front wheels be turned through the same angle, and held rigidly. Then their axes, being parallel, will meet the line of the rear axle, produced, in two distinct points, O, and O' (Fig. 2), and make equal angles, θ, with it.

Now, if the car be made to turn about the point O as a center, the wheel B will skid. Likewise, if the car be made to turn about O' the wheel A will skid. If the car be made to turn about any other point on the line DC, both front wheels will skid. Under these conditions, if the car runs along any curved path whatever, there will be some skidding.

If, however, the inner wheel B is turned through a greater angle than the outer wheel A, so that the points O' and O coincide, then the car will run on a circle with O as center without any skidding whatever. Now in the steering gear of an automobile the cranks are not made parallel to the planes of the front wheels, so that the rod connecting the cranks is shorter than the distance between the two hinge joints. Hence, in the process of steering, the inner wheel is actually turned through a greater angle than the outer wheel.

The object of the following investigation is to determine whether by means of this simple scheme, all skidding can be practically eliminated. If so, what shall be the angle between the crank and the plane of the wheel?

Let $C(a, 0)$ and C' $(-a, 0)$ be the positions of the two pivot joints at the ends of the front axle (Fig. 3).

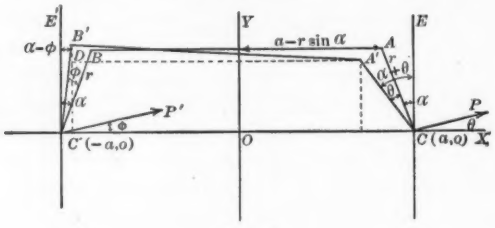


Fig. 3.

Let CE and $C'E'$ be perpendicular to $C'C$, and in the same plane with the cranks CA and $C'B$, of length r. Then CE and $C'E'$ will be horizontal diameters of the front wheels when the car is moving straight forward.

Let $\alpha = \angle ECA = \angle E'C'B$, the constant angles the cranks make with the planes of the wheels.

Let the inner crank CA be turned through an angle θ, to the position CA', then the outer crank $C'B$ will be turned through some angle φ, to the position $C'B'$, making

$$\angle\, ECA' = \alpha + \theta, \qquad \text{and} \qquad \angle\, E'C'B' = \alpha - \varphi.$$

It is required to find the locus of the intersection of the two lines CP and $C'P'$, which are perpendicular to the planes of the wheels, that is, the lines representing the axles of the wheels, produced.

Draw $A'D$ parallel to the x-axis ($C'OC$), meeting the perpendicular from B' on the x-axis, in D. Then

$$A'D^2 + B'D^2 = A'B'^2, \qquad A'D = 2a - r\,[\sin(\alpha+\theta) + \sin(\alpha-\varphi)],$$

$$D\,B' = r\,[\cos(\alpha-\varphi) - \cos(\alpha+\theta)],$$

$$A'B' \equiv AB = 2(a - r\sin\alpha), \text{ a constant;}$$

$$\therefore [2a - r[\sin(\alpha+\theta) + \sin(\alpha-\varphi)]]^2 + r^2[\cos(\alpha-\varphi) - \cos(\alpha+\theta)]^2 = 4(a - r\sin\alpha)^2. \tag{1}$$

This equation reduces to the form

$$2a[\sin(\alpha+\theta) + \sin(\alpha-\varphi)] + r\cos(2\alpha+\theta-\varphi) = r\cos 2\alpha + 4a\sin\alpha, \tag{2}$$

from which the variables θ and φ must be eliminated.

The equations of the two lines CP and $C'P'$ are, respectively,

$$y = (x - a)\tan\theta, \qquad \text{and} \qquad y = (x + a)\tan\varphi,$$

whence

$$\tan\theta = \frac{y}{x-a}, \quad \text{and} \quad \tan\varphi = \frac{y}{x+a}.$$

Let $\lambda_1 \equiv (x-a)^2 + y^2$, and $\lambda_2 \equiv (x+a)^2 + y^2$. Then

$$\sin\theta = \frac{y}{\sqrt{\lambda_1}}, \quad \cos\theta = \frac{x-a}{\sqrt{\lambda_1}}, \quad \sin\varphi = \frac{y}{\sqrt{\lambda_2}}, \quad \cos\varphi = \frac{x+a}{\sqrt{\lambda_2}},$$

$$\sin(\alpha+\theta) = \frac{(x-a)\sin\alpha + y\cos\alpha}{\sqrt{\lambda_1}}, \quad \sin(\alpha-\varphi) = \frac{(x+a)\sin\alpha - y\cos\alpha}{\sqrt{\lambda_2}},$$

$$\sin(\theta-\varphi) = \frac{2ay}{\sqrt{\lambda_1\lambda_2}}, \quad \cos(\theta-\varphi) = \frac{x^2+y^2-a^2}{\sqrt{\lambda_1\lambda_2}},$$

and

$$\cos(2\alpha+\theta-\varphi) = \cos 2\alpha\cos(\theta-\phi) - \sin 2\alpha\sin(\theta-\phi)$$

$$= \frac{(x^2+y^2-a^2)\cos 2\alpha - 2ay\sin 2\alpha}{\sqrt{\lambda_1\lambda_2}}.$$

Substituting these values in (2), and letting $s = \sin\alpha$, $c = \cos\alpha$, $s_2 = \sin 2\alpha$, $c_2 = \cos 2\alpha$, and $k = rc_2 + 4as$, we get

$$2a\left[\frac{s(x-a)+cy}{\sqrt{\lambda_1}} + \frac{s(x+a)-cy}{\sqrt{\lambda_2}}\right] + r\left[\frac{c_2(x^2+y^2-a^2) - 2as_2y}{\sqrt{\lambda_1\lambda_2}}\right] = k, \quad (3)$$

which is the equation of the locus.

Multiplying this equation by $\sqrt{\lambda_1\lambda_2}$ and rationalizing, we get

$$\begin{aligned}
&\{-64a^4(s^2x^2 - c^2y^2 + as_2y - a^2s^2)^2 - 32a^2kr(s^2x^2 - c^2y^2 + as_2y \\
&\quad - a^2s^2)[c_2(x^2+y^2-a^2) - 2as_2y] - 2k^2r^2[c_2(x^2+y^2-a^2) - 2as_2y]^2 \\
&\quad - 8a^2k^2[\lambda_2(sx+cy-as)^2 + \lambda_1(sx-cy+as)^2]\}\lambda_1\lambda_2 + k^4\lambda_1^2\lambda_2^2 \\
&\quad + 16a^4[\lambda_2(sx+cy-as)^2 + \lambda_1(sx-cy+as)^2]^2 + r^4[c_2(x^2+y^2-a^2) \\
&\quad - 2as_2y]^4 - 8a^2r^2[\lambda_2(sx+cy-as)^2 + \lambda_1(sx-cy \\
&\quad + as)^2][c_2(x^2+y^2-a^2) - 2as_2y]^2 = 0.
\end{aligned} \quad (4)$$

On expanding this equation, restoring the values of λ_1 and λ_2, which are both functions of x and y, and also restoring the value of $k \equiv rc_2 + 4as$, we obtain an equation of the 8th degree in which the coefficients of x^8, x^6, x^4, x^2, and the constant term all vanish. The equation can then be divided by $32a^2y$. Hence the x-axis may be regarded as a part of the locus.

Then, after an enormous amount of work in collecting and simplifying the twenty-five coefficients involved, we get the following equation (5) as the final equation of the locus:

(5)

$$
\begin{array}{llllll}
2s^2 & [4a^2s^2 & +4arsc_2 & +r^2c_2^2 & &]x^6y \\
+2s^2 & [4a^2(4s^2-1) & +12arsc_2 & +3r^2c_2^2 & &]x^4y^3 \\
+2s^2 & [4a^2(5s^2-2) & +12arsc_2 & +3r^2c_2^2 & &]x^2y^5 \\
+2s^2 & [4a^2(2s^2-1) & +4arsc_2 & +r^2c_2^2 & &]y^7 \\
-2s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]x^6 \\
+2s_2 & [4a^3(1-3s^2) & -4a^2rs(3-7s^2) & -2ar^2c_2(1-7s^2) & +3r^3sc_2^2 &]x^4y^2 \\
+2s_2 & [4a^3(1-s^2) & -4a^2rs(3-5s^2) & -ar^2c_2(1-10s^2) & +3r^3sc_2^2 &]x^2y^4 \\
+2s_2 & [4a^3s^2 & -4a^2rs(1-s^2) & +2ar^2c_2s^2 & +r^3sc_2^2 &]y^6 \\
-2 & [4a^4s^2 & +4a^3rs(1-7s^2+6s^4) & +a^2r^2(1-16s^2+56s^4-44s^6) & +ar^3sc_2(1-2c_2^2) \quad -r^4s^2c^2c_2^2 &]x^4y \\
+2 & [4a^4(1-8s^2+10s^4) & -8a^3rs(1-9s^2+10s^4) & -a^2r^2(1-18s^2+80s^4-72s^6) & -2ar^3sc_2(1-2c_2^2) \quad +2r^4s^2c^2c_2^2 &]x^2y^3 \\
-2 & [12a^4s^2(1-2s^2) & +4a^3rs(1-11s^2+14s^4) & -2a^2r^2s^2(1-12s^2+14s^4) & +ar^3sc_2(1-2c_2^2) \quad -r^4s^2c^2c_2^2 &]y^5 \\
+6a^2s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]x^4 \\
-as_2 & [8a^4 & -16a^3rs(1-s^2) & -2a^2r^2(3-20s^2+24s^4) & +4ar^3sc_2^2 \quad +r^4c_2(1-2c_2^2) &]x^2y^2 \\
+as_2 & [24a^4s^2 & -8a^3rs(1+s^2) & -4a^2r^2(1-6s^2+6s^4) & +2ar^3sc_2^2 \quad -r^4c_2(1-2c_2^2) &]y^4 \\
+2a^2 & [4a^4s^2(2-3s^2) & +4a^3rs(2-17s^2+18s^4) & +a^2r^2(2-35s^2+124s^4-100s^6) & +2ar^3sc_2'(1-2c_2^2) \quad -2r^4s^2c^2c_2^2 &]x^2y \\
-\tfrac{1}{2}a^2 & [48a^4s^2(1-2s^2) & +16a^3rs(2-19s^2+22s^4) & +4a^2r^2(1-19s^2+84s^4-76s^6) & +8ar^3sc_2(1-2c_2^2) - r^4(1-6s_2{}^2c_2^2)] & y^3 \\
-6a^4s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]x^2 \\
+a^3s_2 & [24a^4s^2 & +8a^3rs(1-5s^2) & -2a^2r^2(1-2s^2-4s^4) & -2ar^3sc_2^2 \quad +r^4c_2(1-2c_2^2) &]y^2 \\
-2a^4 & [4a^4s^2(1-2s^2) & +4a^3rs(1-9s^2+10s^4) & +a^2r^2(1-18s^2+64s^4-52s^6) & +ar^3sc_2(1-2c_2^2) \quad -r^4s^2c^2c_2^2 &]y \\
+2a^6s_2 & [4a^3s^2 & +4a^2rs(1-3s^2) & +ar^2c_2(1-6s^2) & -r^3sc_2^2 &]=0.
\end{array}
$$

Since y is a factor of the seventh degree terms in equation (5), the locus has an asymptote parallel to the x-axis. The equation of this asymptote is found by putting the sum of the x^6y and x^6 terms equal to zero.

This gives

$$y = 2\frac{c}{s}(a - rs) \equiv 2 \cot \alpha(a - r \sin \alpha). \tag{6}$$

Throughout the limited range of motion (variation of θ) that is possible in the automobile, the curve lies very close to this asymptote, and on the same side of it as the x-axis, which is in the position of the front axle.

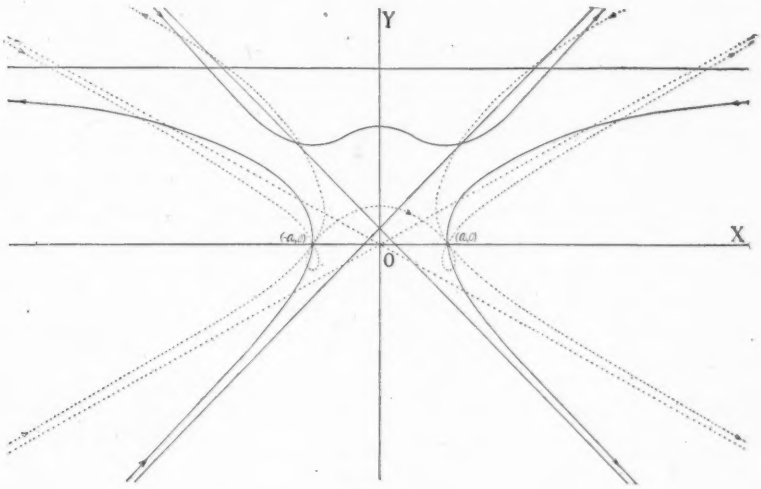


Fig. 4.

For continuous curve, $\alpha = 30°$; for dotted curve, $\alpha = 15°$.

Therefore, if, with given values of a and r, we find a value of α which makes the ordinate of this asymptote a little greater than the wheelbase of the car, there will be very little skidding due to the steering mechanism when the car runs on a curved path.

If $a = 2.1$ ft., $r = \frac{1}{2}$ ft., and $\alpha = 21°$, we get $y = 10.05$ ft.
These would be good values for a car with a wheel-base of 9.5 ft.

It is evident from equation (6) that y increases as α diminishes. Therefore, since a and r may be considered as practically the same in all cars, the greater the wheel-base the smaller the angle α must be.

So far it has been assumed that the cranks are behind the front axle. If the cranks project forward, as in some cars, they must then diverge from the planes of the wheels. In order to get the corresponding equations for this case, it will be sufficient to change the sign of r.

With $a = 2$ and $r = 1$ we get for this asymptote [Eq. (6)] the following equations:

When $\alpha = 30°$, $y = 3\sqrt{3}$. [See Fig. 4].
When $\alpha = 15°$, $y = 13-$. Not shown in Fig. 4.
When $\alpha = 45°$, $y = 2.6$. See dot and dash line in Fig. 5.
When $\alpha = 60°$, $y = 1.3$. See dotted line in Fig. 5.
When $\alpha = 90°$, $y = 0$.
When $\alpha \doteq 0$. $y \doteq \infty$.

If we let

$$l^2 = 4a^2s^2 + 4arsc_2 + r^2c_2^2 = (2as + rc_2)^2,$$

and

$$m^2 = c_2(4a^2 - 4ars - r^2c_2),$$

the 7th degree terms may then be written in the form

$$2s^2y(l^2x^2 - m^2y^2)(x^2 + y^2)^2.$$

Hence there is also a pair of asymptotes parallel to the two lines

$$l^2x^2 - m^2y^2 = 0.$$

After another long process of reduction we find the equations of these asymptotes to be

$$y = \pm\frac{l}{m}x + \frac{as(4a^2s^2 - 4ars^3 - r^2c_2)}{cm^2}. \qquad (7)$$

Let $a = 2$, and $r = 1$.[1] Then for this pair of asymptotes we get the following equations:

When $\alpha = 30°$, $y = \pm\frac{5}{\sqrt{23}}x + \frac{20\sqrt{3}}{69} = \pm x \tan 46°11.'5 + .5$. [See Fig. 4]

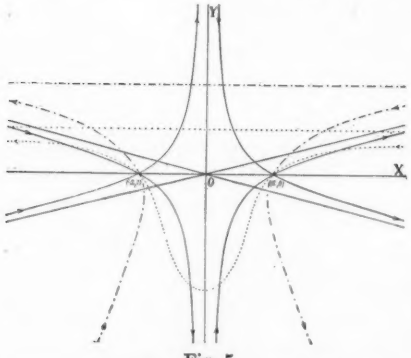


Fig. 5.
For continuous curve $\alpha = 0°$; for dot-dash curve, $\alpha = 45°$; for dotted curve, $\alpha = 60°$.

When $\alpha = 15°$, $y = \pm x \tan 29°-29' + .03$. [See Fig. 4]

When $\alpha = 0$, $x^2 - 15y^2 = 0$. [See Fig. 5]

When $\alpha = 44°59'$,

$$y = \pm x \tan 88°53'.5 + 2948.$$

x-intercepts $= \pm 55.75$.

When $\alpha > 45°$, these asymptotes are imaginary, since m^2 contains the factor c_2, which is then negative.

As $\alpha \doteq 45°$, these asymptotes move off into the infinite region. Hence that part of the curve below the x-axis [Fig. 5] approaches in the limit the form of a parabola with infinite branches extending downward.

[1] All the curves shown in Figs. 4 and 5, were constructed with $a = 2$ in. and $r = 1$ in. Hence the unit of the scale in these figures is half the distance from O to $(a, 0)$.

There are three rather distinct forms of the curve.

(1) When α is less than about 20°. This form is represented by the dotted curve in Fig. 4, which in the limit as $\alpha \doteq 0$, becomes the continuous curve in Fig. 5, the loops below the x-axis then becoming infinite and one part approaching coincidence with the x-axis.

(2) When $45° > \alpha > 20°$. The continuous curve of Fig. 4 is an example of this form, and as $\alpha \doteq 45°$ it becomes the dot and dash curve of Fig. 5, the upper branch in Fig. 4 then passing entirely into the infinite region.

(3) When $\alpha > 45°$. This form is seen in the dotted curve of Fig. 5, and it approaches coincidence with the x-axis as $\alpha \doteq 90°$.

When $\alpha = 0$ the equation of the curve is

$$y^2[4r^2x^4 - 4(4a^2 - r^2)x^2y^2 - 8a^2r^2x^2 + r^2(4a^2 - r^2)y^2 + 4a^4r^2] = 0. \qquad (8)$$

When $\alpha = 45°$ we get

$$\begin{aligned} &2x^2y(x^2 + y^2)^2 - 2(2a - r\sqrt{2})(x^2 - y^2)(x^2 + y^2)^2 \\ &- (4a^2 - 4ar\sqrt{2} + 3r^2)x^4y - 2(2a^2 - 4ar\sqrt{2} + 3r^2)x^2y^3 \\ &+ (4ar\sqrt{2} - 3r^2)y^5 + 6a^2(2a - r\sqrt{2})x^4 - 2a(4a^2 - 2ar\sqrt{2} + r^2)x^2y^2 \\ &+ 2a(6a^2 - 3ar\sqrt{2} + r^2)y^4 + 2a^2(a^2 - 4ar\sqrt{2} + 3r^2)x^2y \\ &+ \tfrac{1}{2}(16a^3r\sqrt{2} - 12a^2r^2 + r^4)y^3 - 6a^4(2a - r\sqrt{2})x^2 \\ &+ 2a^3(6a^2 - 3ar\sqrt{2} + r^2)y^2 + a^4(4ar\sqrt{2} - 3r^2)y + 2a^6(2a - r\sqrt{2}) = 0. \end{aligned} \qquad (9)$$

When $\alpha = 90°$ we get

$$\begin{aligned} &y[(2a - r)^2(x^2 + y^2)^3 - a(2a - r)^2(a + r)x^4 + a(2a - r)(6a^2 - 5ar \\ &+ 2r^2)x^2y^2 + a(12a^3 - 16a^2r + 6ar^2 - r^3)y^4 - a^3(2a - r)^2(a - 2r)x^2 \\ &+ \tfrac{1}{4}a^2(2a - r)(24a^3 - 28a^2r + 6ar^2 - r^3)y^2 + a^5(2a - r)^2(a - r)] = 0. \end{aligned} \qquad (10)$$

Equations (8), (9), and (10) were derived directly from equation (4), and also by substituting in equation (5), and the two results agreed.

The y-intercept of the curve is always $a \tan 2\alpha$.

When α is 30° and 60° these intercepts are $\pm a\sqrt{3}$, respectively. We find that the coördinates $(0, \pm a\sqrt{3})$ satisfy the corresponding equations. This serves as a check on the coefficients of the y terms.

If we let $y = 0$ in equation (5) we get $(x^2 - a^2)^3 = 0$, for all values of α. Hence the coördinates of the points $(\pm a, 0)$ always satisfy equations (5). When α is less than about 20°, the exact value depending upon the relative values of a and r, the curve goes through each of these points three times. These points are then real double points with respect to the "upper branch" of the locus. For larger values of α these points must be considered as conjugate or isolated points, with respect to this branch of the locus.

Thus these points are related to this branch of the locus in the same way as the origin is related to the conchoid $(x^2 + y^2)(x - a)^2 = b^2x^2$.

The coördinates (0, 0) always satisfy this equation, and when $y = 0$, we get $x = 0, 0, a \pm b$, but the curve does not cut the x-axis four times unless b $> a$. That is, the origin is a double point, or a conjugate point according as $b > a$, or $b < a$. The same thing is true in regard to the limaçon

$$(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2).$$

NOTE ON THE INCENTERS OF A QUADRILATERAL.

By F. V. Morley, Johns Hopkins University.

The rectangular configuration of the incenters of an inscribed quadrilateral was known to Professor Neuberg of Liège before 1906.[1] It was rediscovered, as an isolated fact, by the writer in 1914, and several proofs were supplied by his father, Professor Morley. When the Monthly enlarged its scope in 1916 the proposition was sent in as a problem, appearing in March, 1917; the obvious extension and the writer's solution were published soon after.[2] In June, 1918, Professor Altshiller-Court published an article in this Monthly, rediscovering the configuration.[3]

The proposition as an isolated fact with reference to the inscribed quadrilateral was therefore worth discovery; but it will now be shown that the proposition appears in a well-known theory.

I.

There are several beautiful chains of theorems concerned with the elementary geometry of n-lines in a plane. One of these is Clifford's chain, and others of similar type are well known to students of metric or reflexive geometry. With slight modifications, they apply to directed as well as to undirected lines.[4]

One of these chains is concerned with the incenters of n-lines. In the present note it will be simpler to consider the lines as directed. For example, a triangle composed of three undirected lines has four incenters, using the term in its general meaning; but if the three lines are thought of as directed, there is only one circle tangent to all three in the proper sense, and hence only one incenter.

[1] The reference here seems to be to Neuberg's article in *Mathesis*, 1906, pp. 14–17. But Neuberg published the result as a problem many years earlier (*Nouvelle correspondance mathématique*, Tome 1, 1875, p. 96), accompanied by the statement that it was extracted from *Archiv der Mathematik und Physik*, 1842, p. 328. Catalan's solution of the problem appears in *Nouvelle correspondance*, tome 1, pp. 198–200.—Editor.

[2] November, 1917, volume 24, pp. 429–430.

[3] Volume 25, 1918, pp. 241–246; for comment, see volume 26, 1919, pp. 65–66. Still more recently, the subject has been discussed in the comprehensive article by J. W. Clawson, in the *Annals of Math.*, vol. 20, p. 254 (1919). [Mr. Morley's paper was in the hands of the Editor some time before Professor Clawson's paper was published.—Editor.]

[4] Cf. F. Morley, *Transactions of the American Mathematical Society*, Vol. 1, pp. 97–115; and F. H. Loud, *Transactions of the American Mathematical Society*, Vol. 1, pp. 323–338.

Four directed lines may be taken three at a time in four ways; each triangle will have its incenter, and according to the theorem of Steiner which is the basis of the chain, these four incenters lie on a circle.[1] This circle has been called the center-circle[2] of the quadrilateral; its center is a unique point of the quadrilateral, and may be called by analogy its incenter. The notation commonly used is C_4 for the center-circle of four directed lines, and c_4 for the incenter.

The statement of the chain proceeds as follows.[3]

1. *Five directed lines have five points c_4, the incenters of the lines taken four at a time; these five points are on a circle, C_5, whose center, c_5, is the incenter of the 5-line. Six directed lines have six points c_5, on a circle C_6; and in general, n directed lines have n points of the type c_{n-1}, which are on a circle C_n, the center of which is the incenter, c_n, of the n-line.*

2. *Five directed lines have five circles C_4; these circles meet in a point, N_5, called the node of the 5-line; and in general, n directed lines have n circles C_{n-1} on a node N_n.*

These general theorems apply to the circles C_{n-1} and the centers c_{n-1} of n lines. The configuration has associated with it a variety of other circles, C_{n-2}, C_{n-3}, . . . , with corresponding centers, on which there are definite conditions; but the number and complexity of these in general makes their investigation of doubtful value. Special cases occur, however, in which the subcircles and centers simplify, and one of these has recently aroused some interest.

II.

In dealing with four directed lines, the reversal of any one direction will produce an entirely different arrangement of the circles and centers. The same directed quadrilateral will have all the features of five, six, seven, or eight lines, according to the number of its sides which are counted both ways and as the general configuration will be considerably simplified, new properties of the quadrilateral may be emphasized.

Eight directed lines will have 56 points of the type c_3, four at a time on 70 circles C_4. There will be 70 centers c_4, five at a time on 56 circles C_5. The 56 centers c_5 are six at a time on 28 circles C_6; the 28 centers c_6 are seven at a time on 8 circles, C_7 whose centers c_7 form the center-circle of the 8-line. So much the general theorem prescribes; and when the quadrilateral, by reversing its sides, is considered as eight lines, these facts will hold. The centers will double up, however, and there will appear only 28 c_3, 38 c_4, 28 c_5, 16 c_6, 4 c_7; moreover, not all of

[1] Stated without proof in *Annales de mathématiques* (Gergonne), Tome 18, 1828, p. 302. A proof was furnished by J. Mention in *Nouvelles annales de mathématiques*, vol. 21, 1862, p. 16f. The theorem did not, however, originate with Steiner but with L. Puissant; see *Correspondance sur l'Ecole Polytechnique*, Tome 1, 1806, p. 193.—EDITOR.

[2] The term "*center-circle*" was used by F. Morley, *l.c.*, p. 99, for the circle on the circumcenters of four lines taken three at a time. But later, *Trans. Am. Math. Soc.*, vol. 8 (1907), p. 20, he alters the name for the circle on the circumcenters to "*centric-circle.*" Now four directed lines have two distinct circles, one concerned with circumcenters (the centric-circle) and one on the four incenters. This last is the one with which we are concerned, and it is now called the "*center-circle.*" Cf. Hodgson, *Trans. Am. Math. Soc.*, (1912), p. 203.

[3] Loud (*l.c.*), p. 325.

these are distinct, for the centers c_7 coalesce and unite with the incenter of the configuration. The points c_6 will then lie on a circle, and the points c_5 will be affected somewhat.

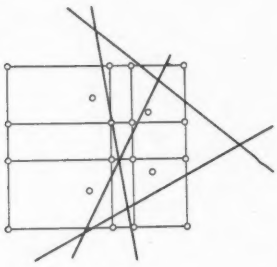


Fig. 1.

It is worth investigation to see how these comparatively few points c_5 behave in the case where the quadrilateral is considered as a special case of eight directed lines; and it is interesting to note in Figure 1 their resemblance to the configuration discussed above; namely, the rectangular net formed by the incenters of an inscribed quadrilateral.

The figure of the circles, C_5, is also of interest in this case, (Figure 2). As their centers are constrained to the rectangular net it is to be expected that they are in families of four; in addition they all pass through a point.

III.

The configuration suggested by the figures is readily proved. It is convenient to consider first a quadrilateral with two sides counted both ways; by repeating this figure the whole configuration may be obtained, but all that is essential is contained in Figure 3.

Here there are six directed lines. By the general theorem there is a center-circle for all six, expressed in the manner of Loud (*l.c.*, p. 325), as

$$x = a_0 - a_1 t$$

where the constants are complex and t an orthogonal number. The six circles C_5 are included in the double infinity of circles

$$x = a_0 - a_1(t + t') + a_2 tt'$$

which are known to pass through a point. Figure 2 is thus confirmed; all the circles C_5 pass through a point, which turns out to be the Clifford point of the quadrilateral.

On examination the six points c_5 in Figure 3 are seen to fall into two sets; four on a rectangle, and two separate. Consider now a triangle with two sides counted both ways; by drawing the centers c_4 it is found that the incenter c_5 coincides with the circumcenter of the triangle. Here there are two such triangles, formed by omitting the single lines 3 and 4 in turn. It follows that the two separate points noticed above are the circumcenters of those triangles, and in passing to the case of Figure 1, the four points c_5 which do not lie on the rectangular net are found to be the circumcenters of the four triangles of the original quadrilateral. They each count for 3 of the c_5, making up with the other 16 the total of 28. It is well known that the circumcenters of a quadrilateral lie on a circle;[1] in this particular case the node or Clifford point is also on this circle.

[1] Steiner, *Gesammelte Werke*, Band 1, p. 323.

That the other four centers in Figure 3 are rectangular, may be proved from the fact that all the circles of which they are centers pass through the node. Through the node there are a pair of natural rectangular axes,[1] indicated in the figure, to which the rectangle is parallel. These axes govern the grouping of the circles in Figure 2.

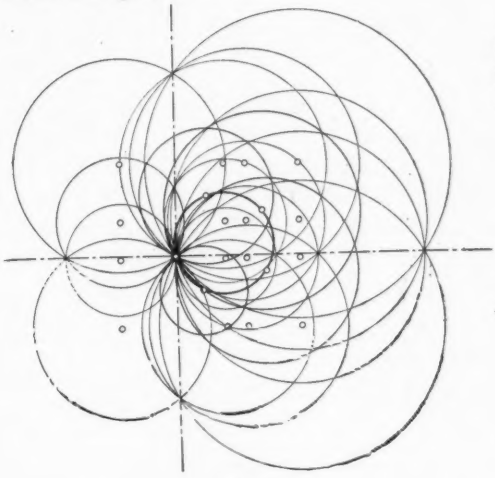


Fig. 2.

In order to identify the rectangular net of Figure 1 with the net of incenters of an inscribed quadrilateral, the 16 points must

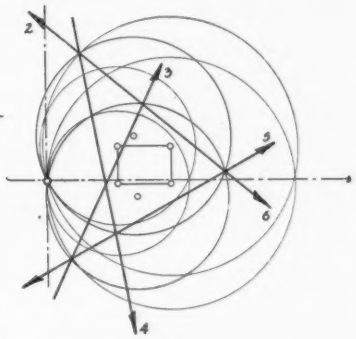


Fig. 3.

be incenters of the four triangles whose centers have been noted as on a circle. To prove this, draw any of the circles C_6, and note that the four points c_5 corresponding to each of the triangles of the inscribed quadrilateral are mutually orthocentric.[2]

The rather pretty configuration resulting from the consideration of the quadrilateral as a special case in the theory of directed lines suggests the investigation of other cases. The centers c_4 of three lines counted both ways, for example, reduce to the nine point arrangement on a circle of Feuerbach. Similar configurations will hold for five or more lines, but the complication will increase. In the general treatment the analytical method of Professor Morley's memoir (*l.c.*), (which is followed in the paper by Dr. Loud (*l.c.*)), involves much less fatigue in the transference of thought than does the geometrical argument as here given.

[1] These are called the *incentric lines* by Clawson, l.c., p. 246.

[2] For a study of the properties of an orthocentric group of points see the article by Professor Altshiller-Court in the last number of this MONTHLY, *On the Orthocentric Quadrilateral*, pp. 199–202. —EDITOR.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

NEW QUESTIONS.

The following interesting questions were formulated by Professor S. Kakeya at the request of the editors, and transmitted through Professor Birkhoff. Professor Kakeya has treated some problems of similar nature in various papers in the *Science Reports* of the Tôhoku Imperial University. They involve considerations of decided difficulty in the analytic formulation of geometric relations.

39. There are certain problems in geometry which are simple in statement but can be reduced only to very complicated problems in transcendental analysis. Following are several examples of the type of problem in question.

1. What is the smallest plane area within which a given figure can be turned through a complete revolution? It is not implied that the figure should revolve about a fixed point, but merely that in the course of its motion the figure should have every possible orientation in the plane. The problem may be modified by considering only convex areas.

An interesting special case is that in which the given figure is a segment of a straight line. In this case it has been conjectured by Professors Osgood and Kubota that the smallest area may be bounded by a three-cusped hypocycloid; if we consider only convex areas, perhaps the result will be an equilateral triangle. I have no indication of a proof.

2. For every closed convex curve of area P there is an n-sided circumscribed polygon of least area Q and an n-sided inscribed polygon of greatest area R. For a fixed value of the integer n and for all closed convex curves, what is the upper limit of Q/P and what is the lower limit of R/P? I have succeeded only in proving that for the case $n = 3$, the upper limit of Q/P is 2.

3. Let the area of a given simple closed curve A be a. Remove from A the greatest possible area a_1 similar to another given simple closed curve B. From the remaining figure remove the greatest possible area a_2 similar to B. Continue this process indefinitely. Is it or is it not true that

$$a_1 + a_2 + a_3 + \ldots = a?$$

I have proved the statement to be true in the special case that A is convex and B is a circle.

4. Let a given closed convex curve K have the property that a given triangle whose angles are incommensurable with π can be revolved completely within K (see part 1 of this question), always remaining inscribed in K. What may the curve K be? Can any other curve except a circle satisfy the conditions?

DISCUSSIONS.

For a special type of quintic equation whose coefficients involve two parameters, Mr. C. B. Haldeman, in the first discussion following, obtains an algebraic solution by means of elementary operations and the extraction of roots; he also derives expressions for the roots in trigonometric form, and gives a graphical construction by the use of a certain cubic curve and a circle.

Professor Trevor, who has previously given instances of the use of mathematics in thermodynamics (*1919*, 444–447; *1920*, 55–57), presents an interesting set of results in connection with the study of a state of thermodynamic equilibrium in a homogeneous fluid. Several transformations of independent variables in a fundamental relation yield corresponding interpretations in the physical problem.

Professor Ransom gives a simple formula which may be used as a check in connection with the ambiguous case in the solution of plane triangles. Of course it applies only to problems which yield two solutions. Apparently there is a similar formula for the analogous case of spherical triangles:

$$\tan \tfrac{1}{2}(c' + c'') = \tan b \cos A,$$

and another for the dual case of ambiguity.

Professor Dick discusses a relationship between the dimensions of the "King's Chamber" in the Great Pyramid, the regular pentagon, and the regular icosahedron and dodecahedron. The implication of a connection between this relationship, the occurrence of the numbers of Fibonacci in plant life, and certain mystic statements of the ancient mathematicians will perhaps seem to most readers to be somewhat far-fetched. It should also be stated that the relation between a regular pentagon and the inscription of a square in a semicircle is clearly implied by the usual construction.

I. Resolution of a Certain Quintic Equation and a Geometrical Construction for its Roots.

By C. B. Haldeman, Ross, Butler County, Ohio.

1. The transformation

$$y = x - \frac{a}{x}$$

will reduce the equation $y^5 + 5ay^3 + 5a^2y + 2b = 0$ to $x^{10} + 2bx^5 - a^5 = 0$; from which we find x, and consequently

$$y = \sqrt[5]{-b + \sqrt{b^2 + a^5}} + \sqrt[5]{-b - \sqrt{b^2 + a^5}}.$$

If $b^2 + a^5$ be negative it appears the roots of the given equation are trigonometrical functions of the coefficients; for the transformation

$$y = -2S\sqrt{-a}$$

will reduce the given equation to

$$16S^5 - 20S^3 + 5S = \frac{b}{a^2\sqrt{-a}};$$

and since

$$16\sin^5 A - 20\sin^3 A + 5\sin A = \sin 5A,$$

we may take

$$S = \sin A, \qquad \frac{b}{a^2\sqrt{-a}} = \sin 5A$$

and get

$$y = -2\sqrt{-a}\sin \tfrac{1}{5}\sin^{-1}\frac{b}{a^2\sqrt{-a}},$$

$$y = -2\sqrt{-a}\sin\frac{1}{5}\left(2\pi + \sin^{-1}\frac{b}{a^2\sqrt{-a}}\right),$$

$$y = -2\sqrt{-a}\sin\frac{1}{5}\left(\pi - \sin^{-1}\frac{b}{a^2\sqrt{-a}}\right),$$

$$y = 2\sqrt{-a}\sin\frac{1}{5}\left(\pi + \sin^{-1}\frac{b}{a^2\sqrt{-a}}\right),$$

$$y = 2\sqrt{-a}\sin\frac{1}{5}\left(2\pi - \sin^{-1}\frac{b}{a^2\sqrt{-a}}\right).$$

2. The equation

$$xy\sqrt{-(b^2+a^5)} = a^2y^3 + by^2 + 3a^3y + 2ab \tag{1}$$

represents a real trident of Newton[1] when $b^2 + a^5$ is negative, and

$$x^2 + y^2 = -4a \tag{2}$$

is the equation of a circle, which will be real when a is negative. Eliminating x from (1) by means of (2), the result may be placed under the form

$$(y^5 + 5ay^3 + 5a^2y + 2b)\,(a^2y + 2b) = 0.$$

The first of these factors is identical with the quintic expression given above and has five real roots when $b^2 + a^5$ is negative. From this it appears the five real roots may be represented by the ordinates of the intersections of a trident and a circle. The five intersections, whose ordinates are the five real roots of this equation, are the vertices of a regular pentagon inscribed in a circle whose radius is $2\sqrt{-a}$, as may be seen by reference to the above values of y.

II. Certain Mathematical Features of Thermodynamics.

By J. E. Trevor, Cornell University.

Let e, v, s denote the energy, volume, and entropy of unit mass of a body of homogeneous fluid in a state of thermodynamic equilibrium under the pressure p at the te.nperature θ. The energy e is then a continuous function of v, s, and equilibrium subsists when and only when

$$de = -pdv + \theta ds. \tag{1}$$

Hence the conditions of equilibrium are the equations

$$-p = \partial e/\partial v, \qquad \theta = \partial e/\partial s. \tag{2}$$

The state of equilibrium is stable, with respect to small displacements, when

[1] One of the four canonical forms (no. 108) to which Newton has reduced the general equation of cubic curves in his *Enumeratio linearum tertii ordinis*, first printed in Newton's *Optics*, London, 1704.—Editor.

and only when all possible sets of variations δe, δv, δs of the energy, volume, and entropy of the body satisfy the inequality

$$\delta e + p\delta v - \theta\delta s > 0.$$

Hence the necessary and sufficient conditions of stability are

$$\frac{\partial^2 e}{\partial v^2}\frac{\partial^2 e}{\partial s^2} - \left(\frac{\partial^2 e}{\partial v \partial s}\right)^2 > 0, \qquad \frac{\partial^2 e}{\partial v^2} > 0.$$

A consequence is that $\partial^2 e/\partial s^2 > 0$.

For various purposes it is desirable to employ v, θ or p, s or p, θ as independent variables, the remaining three of the variables e, p, v, θ, s then being functions of those chosen. The determinants of these transformations are

$$\frac{\partial(v, \theta)}{\partial(v, s)} = \frac{\partial^2 e}{\partial s^2}, \qquad \frac{\partial(p, s)}{\partial(v, s)} = -\frac{\partial^2 e}{\partial v^2}, \qquad \frac{\partial(p, \theta)}{\partial(v, s)} = -\frac{\partial^2 e}{\partial v^2}\frac{\partial^2 e}{\partial s^2} + \left(\frac{\partial^2 e}{\partial v \partial s}\right)^2,$$

and by the conditions of stability none of these determinants vanish. When functions of the new variables are defined by the equations

$$f(v, \theta) = e - \theta s, \qquad g(p, s) = e + pv, \qquad h(p, \theta) = e + pv - \theta s,$$

differentiation and comparison with (1) yields the criterion of equilibrium in the forms

$$(3) \qquad df = -pdv - sd\theta, \qquad dg = vdp + \theta ds, \qquad dh = vdp - sd\theta.$$

The conditions of integrability of the second members of (1) and (3) are Maxwell's "thermodynamic relations,"

$$(4) \qquad -\frac{\partial p}{\partial s} = \frac{\partial \theta}{\partial v}, \qquad \frac{\partial p}{\partial \theta} = \frac{\partial s}{\partial v}, \qquad \frac{\partial v}{\partial s} = \frac{\partial \theta}{\partial p}, \qquad \frac{\partial v}{\partial \theta} = -\frac{\partial s}{\partial p}.$$

It may be asked whether p, v and θ, s can serve as sets of independent variables. We have that

$$-\frac{\partial(v, p)}{\partial(v, s)} = \frac{\partial(\theta, s)}{\partial(v, s)} = \frac{\partial^2 e}{\partial v \partial s};$$

and, as discussed below, it may be taken to be a fact of observation that these determinants vanish, if at all, only at points on a line $\partial^2 e/\partial v^2 = 0$. The circumstance that any one of the variables p, v, θ, s is a function of any two of the others gives rise to the consideration of six pairs of variables that can be taken successively independent, of twelve functions of pairs of variables, and of a set of twenty-four first derivatives of these functions. The derivatives of the set may conveniently be assigned to two classes, Class I containing the derivatives of the "work variables" p, v, and of the "heat variables" θ, s, with regard to each other, and Class II containing the derivatives of a work variable with regard to a heat variable, and the reverse. Making use of the thermodynamic relations (4)

connecting the cross derivatives, the proposed classification is exhibited in the following table:

Class I	Class II
$-\left(\frac{\partial p}{\partial v}\right)_\theta = \frac{\Delta}{e_{22}}$	$-\left(\frac{\partial p}{\partial s}\right)_v = +\left(\frac{\partial \theta}{\partial v}\right)_s = \frac{e_{12}}{1} = e_{12}$
$-\left(\frac{\partial p}{\partial v}\right)_s = \frac{e_{11}}{1}$	$-\left(\frac{\partial p}{\partial \theta}\right)_v = -\left(\frac{\partial s}{\partial v}\right)_\theta = \frac{e_{12}}{e_{22}} = f_{12}$
$+\left(\frac{\partial \theta}{\partial s}\right)_p = \frac{\Delta}{e_{11}}$	$-\left(\frac{\partial v}{\partial s}\right)_p = -\left(\frac{\partial \theta}{\partial p}\right)_s = \frac{e_{12}}{e_{11}} = -g_{12}$
$+\left(\frac{\partial \theta}{\partial s}\right)_v = \frac{e_{22}}{1}$	$-\left(\frac{\partial v}{\partial \theta}\right)_p = +\left(\frac{\partial s}{\partial p}\right)_\theta = \frac{e_{12}}{\Delta} = -h_{12}$

Using the notation $e_{11} = \partial^2 e/\partial v^2$, $e_{12} = \partial^2 e/\partial v \partial s$, $e_{22} = \partial^2 e/\partial s^2$, $\Delta = e_{11}e_{22} - e_{12}^2$, the tabulated values of the derivatives in terms of second derivatives of $e(v, s)$ are found from the equations

$$-dp = e_{11}dv + e_{12}ds,\ d\theta = e_{12}dv + e_{22}ds,$$

obtained by differentiating the conditions of equilibrium (2). The derivatives of Class II have the values e_{12}, f_{12}, $-g_{12}$, $-h_{12}$; *i.e.*, they form the terms of the conditions of integrability of the second members of the equations (1) and (3). The tabulated derivatives and their reciprocals constitute the set of twenty-four derivatives. By the "reciprocal" of $(\partial p/\partial v)_\theta = -\Delta/e_{22}$, for example, is meant the derivative $(\partial v/\partial p)_\theta = -e_{22}/\Delta$.

The signs of the twenty-four rates of change of the quantities p, v, θ, s with regard to one another afford extensive information concerning the thermodynamic properties of fluids. By the conditions of stability we have that, for all realizable states of equilibrium, e_{11}, e_{22}, Δ are positive. It appears then that, for realizable states of equilibrium, the signs of the derivatives of Class I are determined by the conditions of stability, while the signs of the derivatives of Class II are determined by the sign of e_{12}, which must be found by observation. From $(\partial v/\partial \theta)_p = -e_{12}/\Delta$ we observe that e_{12} is negative when the specific volume rises on heating at constant pressure, and that e_{12} changes sign if the representative point crosses a line of maximum density at constant pressure. It is clear that the fluid states ordinarily observed lie in a field A (in which $e_{12}<0$), but that a field B (in which $e_{12}>0$) appears if, as in the case with liquid water, a locus of maximum density occurs. The derivatives whose values (having regard to their signs) are $e_{12}, f_{12}, -g_{12}, -h_{12}$ are negative in the field A, positive in the field B, and zero on the "zero line" $e_{12} = 0$. A moment's reflection serves to refer a given derivative, with the proper sign, to e_{12} or f_{12}, etc. When "heating" is understood to mean increase of entropy or rise of temperature, and "expansion" means increase of volume or fall of pressure, the circumstance that e_{12} is positive

at all points in the field B requires that, in this field, *heating at constant volume decreases the pressure, and heating at constant pressure decreases the volume, while isentropic expansion increases the temperature, and isothermal expansion develops heat.* In the field A the contrary occurs.

Many interesting conclusions may be drawn. For example, by the definitions of the specific heats at constant volume and at constant pressure, and by reference to the table,

$$c_p = \theta\left(\frac{\partial s}{\partial \theta}\right)_p = \theta\frac{e_{11}}{\Delta}, \qquad c_v = \theta\left(\frac{\partial s}{\partial \theta}\right)_v = \frac{\theta}{e_{22}};$$

whence, by subtraction,

$$\frac{c_p - c_v}{\theta} = \frac{e_{11}}{\Delta} - \frac{1}{e_{22}} = \frac{e_{12}^2}{e_{22}\Delta}.$$

It follows that, $c_p > c_v$ everywhere except at points on the zero line, where $c_p = c_v$.

Again, choosing v, θ as independent variables, let us consider the specific heat c of a fluid with regard to an arbitrary assigned path $\phi(v, \theta) = 0$. The "heat-differential" θds is

$$\theta\frac{\partial s}{\partial \theta}d\theta + \theta\frac{\partial s}{\partial v}dv;$$

wherefore, on dividing by $d\theta$ and using the second thermodynamic relation and the definition of c_v,

$$c = c_v + \theta\frac{\partial p}{\partial \theta}\frac{dv}{d\theta},$$

where $dv/d\theta$ is the slope of the path. The quantity $\partial p/\partial\theta$, which is equal to $-f_{12}$, is positive in the field A and negative in the field B. Hence at any point in A the specific heat c is an increasing linear function of $dv/d\theta$, while at any point in B it is a decreasing linear function of this slope. Its value at a given point depends only on the *slope* of the path. At any point on the zero line $\partial p/\partial\theta = 0$, wherefore c is *constant* for all finite values of $dv/d\theta$. In particular $c_p = c_v$, as found above. When $dv/d\theta$ is infinite the value of c must be found by evaluating an indeterminate form. Two paths for which $dv/d\theta$ is infinite are the isotherm, for which $c = \infty$, and the isentropic, for which $c = 0$.

A graphic representation of the values of the general specific heat at points on the zero line is had when paths through the point are drawn on the surface representing the region of fluid states in the v, s, θ — space. In this surface the zero line is the locus of the minimum points of isentropic section of a trough. For at points on the zero line $(\partial\theta/\partial v)_s = e_{12} = 0$.

When the path is the curve of section of the surface by a cylinder (or plane) perpendicular to the s, θ-plane and cutting the zero line, its projection on the v, θ-plane has a horizontal tangent at the zero line. Its projection on the s, θ-plane is a curve with two coincident branches, whose slope $ds/d\theta$ at the point of meeting the zero line determines the value, at this point, of the specific heat $c = \theta \cdot ds/d\theta$ with regard to the path. The isotherm, for which $ds/d\theta = \infty$, and the isentropic, for which $ds/d\theta = 0$, are such paths.

When the path is any other curve whose projection on the v, θ-plane has a horizontal tangent at the zero line, its projection on the s, θ-plane is a curve of two distinct branches that meet the zero line in a cusp. The common slope $ds/d\theta$ of these branches at the point of meeting determines the value, at this point, of the specific heat c with regard to the path.

When the path is any curve whose projection on the v, θ-plane has not a horizontal tangent at the point of meeting the zero line, its projection on the s, θ-plane is a curve tangent to the zero line at the point of meeting. Hence the slope $ds/d\theta$ of the zero line at this point determines the value there of the specific heat c with regard to the path. The isobar and the isometric (the curve of constant volume) through the point are such paths.

III. A Check Formula for the Ambiguous Case in Plane Triangles.

By W. R. Ransom, Tufts College.

In the solution of a triangle for which sides a and b and angle A are given, two values B' and B'' are first obtained for the angle opposite b; then two angles C' and C'' are found, and finally two sides c' and c''. The obvious relation $\frac{1}{2}(c' + c'') = b \cos A$ may be used to discover the presence of an error in either of the two triangles that have been computed. This formula does not appear in any text book with which I am acquainted: has it not been employed by some one?

IV. The "King's Chamber" and the Geometry of the Sphere.

By F. J. Dick, Râja-Yoga College.

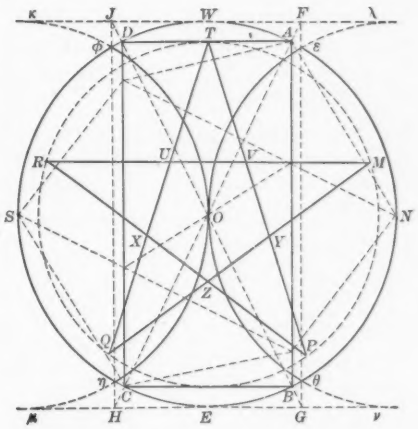
That the designers of the Great Pyramid possessed a thorough knowledge of the geometry of the sphere has been recognized by some, although the usual view[1] is confined to the recognition of their knowledge of the value of π. The length of the "King's Chamber" is exactly double the breadth, while its height is exactly half the diagonal of the floor.[2] Thus if the width be called 2, and the length 4, the "cubic diagonal" of the chamber is 5.

Attention is drawn to the significance of this in connexion with the geometry of the sphere. Let the rectangle $DABC$, 4 units by 2, represent the floor plan (a shape, by the way, found in many ancient temples). Let the circumscribed circle represent the diametral section of a sphere and let two other spheres touch at the center as shown forming the double *vesica piscis* $\phi O \eta S$ and $\epsilon N \theta O$, determining the planes JH and FG cutting the cylindrical envelope $\kappa\lambda\mu\nu$. The diagonals JG and FH coincide with the diagonals of $DABC$, and are each 5 in length, *i.e.*, they are of the length of the cubic diagonal of the "King's Chamber." These are the traces of cones cutting the sphere $WNES$ in the small circles whose diameters are AB, CD. Join AN, and draw the circle $TMPQR$. Then AN measures a side of the pentagon $TMPQR$ whose diagonals only are shown. Pro-

[1] W. M. F. Petrie, *The Pyramids and Temples of Gizeh*, London, 1883.
[2] *Op. cit.*, p. 195.

jecting this and its reflex on AB, CD we have of course the projection of the icosahedron on the plane containing two opposite edges AN, SC, while $UVYZX$ is one face of the internal dodecahedron formed by joining the 12 angular points of the icosahedron.

This diagram appears to show a relation between the *vesica piscis* and the "length-double-the-width" features of some archaic architecture. Incidentally it shows that the inscription of a square in a semicircle places the icosahedron in the sphere at one stroke, so to say.



Taking AN or TM, an edge of the icosahedron, as unity, the edge of the internal dodecahedron is represented by the limit of the sequence

$$\frac{1}{2}, \frac{1}{3}, \frac{2}{5}, \frac{3}{8}, \frac{5}{13}, \frac{8}{21}, \frac{13}{34}, \ldots,$$

the elements of which are formed in obvious fashion from the elements of the series of Fibonacci.[1] This series is found in nature. In certain species of plants, the denominators give the number of shoots or twigs corresponding to a number of spiral circuits given by the numerator.

It seems just possible that the geometers of ancient Egypt who, like the later Pythagorean and Platonic schools, derived their knowledge from ancient Âryâvarta, knew well what they meant when suggesting that the world-universe was built on number and the geometry of the dodecahedron. And it may be that we possess the merest fragments of what was actually taught in the temples of old. But we do have some of their mighty works in stone. Have they been read and fully understood?

RECENT PUBLICATIONS.

REVIEWS.

Euclid in Greek. Book I, with Introduction and Notes. By SIR THOMAS L. HEATH. Cambridge, 1920. Pp. x + 240. Price 10s.

Nearly ten years ago Sir George Greenhill, sitting at his baize-covered work table in Staple Inn, Holborn, in an old-world library well known to many scholars from many lands, made the remark to a visitor from over seas that he felt that the only way to teach plane geometry was by a study of Euclid in the original Greek. The remark led to an interesting discussion upon the present state of

[1] See the article by R. C. Archibald in this MONTHLY for May, 1918, vol. 25, p. 235.

education, with the result that the distinguished scholar took the ground that his opinion was by no means chimerical; while his visitor, although lamenting the present debasement of scholarly ideals on the part of certain educational leaders, maintained that such a plan was hopeless of accomplishment in the present generation.

And now, of all times, when the world seems bent solely upon selfish class activity, when every group seems determined to profit by the misfortunes of every other group, and when idealism seems to have died the death of the martyr, there comes like a voice from another sphere one of the most interesting little volumes that has appeared from the educational press in many a year,—scholarly in composition, delightful in style, and dignified and artistic in its typographical features. When one thinks of the state of mind in which Europe finds herself at present, the book is a surprise; when he looks at the other books which have recently come from her various presses, with their poor paper and poor press work, the feeling of mere surprise becomes one of pleased astonishment.

Anticipating the protest of οἱ πολλοί in the group of modern educators Sir Thomas faces the issue very frankly in his preface:

"In these days when Greek is supposed to be on its trial and Euclid happily defunct, it may well seem a wildly reactionary proceeding to suggest to teachers a combination of the two, a piling (so it might be thought) of one inutility on another. But, first, we must bear in mind that it is only compulsory Greek that is threatened: when that is gone, the study of Greek will be no whit less necessary to a complete education. Generation after generation of men and women will still have to go to school to the Greeks for the things in which they are our masters; and for this purpose they must continue to learn Greek. Again, Euclid can never at any time be more than apparently in abeyance; he is immortal. Elementary geometry will also continue to form part of a complete education; and elementary geometry *is* Euclid, however much the editors of text-books may try to obscure the fact."

Such words may properly be looked upon as food for thought, not merely in England and in Europe generally, but in America as well. Surely the reign of the educational destructionists must be nearly over in this country,—that is, of the young and vigorous group of those who destroy without rebuilding, setting individual tastes above all world experience, and unceasingly talking about social uplift while encouraging in every possible way all intellectual debasement. If this reign is in reality drawing to a close, as indications seem happily to show, and if we are about to rebuild our educational structure, then the question may very properly arise as to whether elective Greek may not find a welcome in our newer type of progressive senior high schools.

If such should be the case, there might well replace some of the literary classics studied in the past that scientific classic, read more widely than any of the other great pieces of literature that Greece produced, published in more editions than all the other scientific works of the ancient world combined, namely, the Elements of Euclid, of which Sir Thomas Heath has already given us the best complete

edition and of which he now gives us Book I in a form adapted to the needs of the schools.

What would be the effect upon the pupil of studying Euclid I in the original Greek, not as a required task but as an elective subject? It would undoubtedly be two-fold in its nature. In the first place, Greek would have a new significance. In general this language has been studied in this country largely as a piece of linguistic gymnastics, and that is one of the reasons why it has so sadly fallen. If it had been taught for reading purposes, for the sake of knowing how the Greeks thought and acted, and for the influence which it should exert in the use of our English tongue, it would never have fallen to its present low estate. If it could, even for a short time, be shown to apply to another line of intellectual activity, one more closely related to another subject in the curriculum, there can be no doubt that students would look at it in quite a different light.

In the second place, geometry would have new significance and a more potent striking force. Those facts which we acquire by serious work are the ones that stay with us. The current theories of tabloid education, whereby we get the thought of Greece, of Rome, and even of the foreign lands of to-day in condensed translations, are too unpsychological to stand for long. In the realm of geometry the case is not the same, since this science still requires some vigor of thought; but the study of a single book of Euclid in the Greek, supplementing the regular course in mathematics, could not fail to impress the meaning of the science upon the pupil's mind and to show him the dignity and logic of geometry at its best. That this will be at all general in American schools is not to be expected, at least until we get a new generation of educators, but that it could be tried with profit in those schools that still emphasize scholarship would seem to be evident.

The work consists of three parts, (1) the introduction, 40 pages; (2) the Greek text of Book I, 70 pages; and (3) the notes, 120 pages. The introduction, written in a style that will appeal to students as well as teachers, contains a brief biography of Euclid, a résumé of the history of geometry in Greece, a sketch of the later history of the Elements, particularly among the Arabs and in medieval Europe, and finally a brief statement of the position of Euclid in the domain of education since the making of the first translations into Latin.

The text itself is clearly printed,—much better indeed, than the English version in such school editions as the familiar ones of Todhunter and Simson.

It is in the notes, however, that the student, the teacher, and the general reader will find the editor at his best. No other living writer can so skillfully interpret to the English reader the finer shades of meaning of the Greek mathematicians, and to the teacher of elementary geometry it will be an inspiration to read this critical study of the first part of the greatest textbook on the subject, all the circumstances considered, that the world has ever produced. To the teacher who has never critically considered, for example, the definitions of straight line, plane, angle, and circle; or who feels that he has come upon an epoch-making discovery that postulates and axioms are essentially the same, or that the Greeks so considered them; or who thinks that he rivals Euclid by finding some new

sequence for his propositions or some new foundation on which to build,—to such a teacher these notes will seem like the words of one having authority and not as those of the educational scribes and Pharisees.

Not least among the valuable features of the work is the index of Greek terms and the index of proper names, aids which readers so often miss in books of this general nature.

In America the book will serve an immediate purpose, in that it is one of the few books on geometry that no teacher can afford to be without, that is indispensable in the library of any well-equipped high school, and that the general reader with scholarly taste will welcome as a pleasant relief from most of our current educational literature. But it is also to be hoped that it will serve still another purpose, the one already referred to as supplying a new classic for those elective courses which may very likely come with the development of a better and more modern type of senior high school in this country.

DAVID EUGENE SMITH.

Theory of Maxima and Minima. By HARRIS HANCOCK. Boston, Ginn and Company, 1917. xiv + 194 pages. Octavo. Price $2.50.

The little treatise deals with maxima and minima, the first half being concerned principally with the case of functions of two variables. The first chapter disposes effectually of the one dimensional problem, and some sections treat in particular examples of three independent variables. The second half, chapters V to VIII, treat the general case. Ordinary and extraordinary maxima and minima, the latter occurring only at irregular points of the function are considered, and considerable space is given relative extremes in addition to the usual discussion of absolute extremes. The discussion treats the problem from many points of view and is particularly rich in elements common to other mathematical disciplines.

The principal topics emphasized are homogeneous forms, definite, semi-definite, and indefinite, and their relation to the determination of maxima and minima; the fallacy of the Lagrangian criterion by which the point in question is approached on straight lines only; various attempts to improve the theory particularly those of Stolz, Scheefer, and v. Dantscher; homogeneous quadratic forms in many variables; the treatment of conditions by obvious and also by more symmetric methods; applications to geometrical and physical problems; subsidiary conditions as inequalities rather than equations of condition; Gauss's principle in problems of mechanics; reversion of series; certain fundamental conceptions in the theory of analytic functions, such as analytic and algebraic dependence, and algebraic singularities.

The readers of the MONTHLY are doubtless most interested in the question as to the place in a college curriculum this book may be expected to fill. The author mentions in his preface, "As introductory to a courese of lecture on the calculus of variations, I have for a number of years given a brief outline of the

theory of maxima and minima." It must not be supposed that the book under discussion depends for its completeness on any subsequent course in the calculus of variations. It should be regarded rather as a self-contained exposition of an important elementary subject too seldom mentioned as an independent discipline but one which despite its present fairly obvious form, has been a subject of error on the part of many of the most conspicuous modern mathematicians, such as Lagrange, Bertrand, Serret, etc. The treatment is made concrete by numerous numerical examples of no little interest on their own account, and of the sort which usually appeal to undergraduate students. The reviewer regards the book as a very eligible candidate for a short post-calculus course. It does not require the acumen essential in an efficient course in real variables nor does it demand as much time as a fair course in elementary differential equations, while it shares with the former the chance of developing the critical faculties of the student and with the latter the numerical concreteness desired by him.

If the calculus of variations is to be taught, such an elementary view as given in Byerly's little course *An Introduction to the Calculus of Variations*, 1917, needs no book of the sort under review. On the other hand, Bolza's well-known *Lectures on the Calculus of Variations* presupposes a familiarity with analysis, and particularly with the notions of real variables for which the present book is in no sense a sufficient introduction. The author's remark "a treatment of these cases, the extraordinary cases, required more study than was anticipated," merely illustrates the fact that the treatment of the case of a finite number of variables involves difficulties of its own and is not to be viewed as a mere preface to the calculus of variations. That these difficulties are not unworthy of attention the book amply demonstrates.

The reviewer noted several features which he regards as unfortunate in a treatment of this sort, of which the most important are as follows. The student is required to have a precise familiarity with the notion of continuity, and the idea of uniform continuity is in essence involved many times but no word of caution is given to such students as may be somewhat hazy as to the exact definition. While the theorems are in general given neither name nor number, we note "Stolz's added theorem," "The Scheefer theorem otherwise stated" (which fills nearly a page), as unfortunate names for fundamental theorems. "A function $f(x)$ is *regular* in the neighborhood of the position $x = a$, if the function in this neighborhood has everywhere a definite value which changes in a continuous manner with x," p. 73, and here real variables only are under consideration! "Legendre was able indeed to show that this sum could not be greater than two right angles; however he did not show *they* could not be less than two right angles," p. 135, italics not in the text. "If a continuous variable quantity is defined in any manner this quantity has an *upper* and a *lower* limit; that is, there is a definitely determined quantity g of such a kind that no value of the variable can be greater than g, although there is a value of the variable

which can come as near to g as we wish . . .," p. 136, while the character of the region of definition is treated as wholly arbitrary.

ALBERT A. BENNETT.

Jahrbuch über die Fortschritte der Mathematik . . . herausgegeben. Von E. LAMPE† und A. KORN. Band 45. Jahrgang 1914-1915. (In 3 Heften) Heft 1. Berlin und Leipzig, Vereinigung Wissenschalflicher Verleger, 1919, 12+368 pp.

The Heft opens with a fine portrait and a seven page appreciation of Emil Lampe's life and work. He was an editor of the "*Fortschritte*" since Jahrgang 1883. The Heft covers History and philosophy, Algebra, Arithmetic, and about twenty five pages of the fourth section on Combinatory analysis and the calculus of probabilities. The number of pages for the first three sections is about eighty more than for the corresponding sections of Jahrgang 1913, and about sixty more than for a similar portion of Jahrgang 1912.

The Theory of the Imaginary in Geometry together with the Trigonometry of the Imaginary. By J. L. S. HATTON, Cambridge, at the University Press, 1920. Royal 8vo. 8 + 216 pp. Price 18 shillings.

Preface: "The position of any real point in space may be determined by means of three real coördinates, and any three real quantities may be regarded as determining the position of such a point. In geometry as in other branches of pure mathematics the question naturally arises, whether the quantities concerned need necessarily be real. What, it may be asked, is the nature of the geometry in which the coördinates of any point may be complex quantities of the form $x + ix'$, $y + iy'$, $z + iz'$? Such a geometry contains as a particular case the Geometry of real points. From it the geometry of real points may be deduced (*a*) by regarding x', y', z' as zero, (*b*) by regarding x, y, z as zero, or (*c*) by considering only those points, the coördinates of which are real multiples of the same complex quantity $a + ib$. The relationship of the more generalized conception of geometry and of space to the particular case of real geometry is of importance, as points, whose determining elements are complex quantities, arise both in coördinate and in projective geometry.

"In this book an attempt has been made to work out and determine this relationship. Either of two methods might have been adopted. It would have been possible to lay down certain axioms and premises and to have developed a general theory therefrom. This has been done by other authors. The alternative method, which has been employed here, is to add to the axioms of real geometry certain additional assumptions. From these, by means of the methods and principles of real Geometry, an extension of the existing ideas and conceptions of geometry can be obtained. In this way the reader is able to approach the simpler and more concrete theorems in the first instance, and step by step the well-known theorems are extended and generalized. A conception of the imaginary is thus gradually built up and the relationship between the imaginary and the real is exemplified and developed. The theory as here set forth may be regarded from the analytical point of view as an exposition of the oft quoted but seldom explained 'Principle of Continuity.'

"The fundamental definition of Imaginary points is that given by Dr. Karl v. Staudt in his *Beiträge zur Geometrie der Lage;* Nuremberg, 1856 and 1860. The idea of (α, β) figures, independently evolved by the author, is due to J. V. Poncelet, who published it in his *Traité des Propriétés Projectives des Figures* in 1822. The matter contained in four or five pages of Chapter II is taken from the lectures delivered by the late Professor Esson, F.R.S., Savilian professor of geometry in the University of Oxford, and may be partly traced to the writings of v. Staudt. For the remainder of the book the author must take the responsibility. Inaccuracies and inconsistencies may have crept in, but long experience has taught him that these will be found to be due to his own deficiencies and not to fundamental defects in the theory. Those who approach

the subject with an open mind will, it is believed, find in these pages a consistent and natural theory of the imaginary. Many problems however still require to be worked out and the subject offers a wide field for further investigations."

Contents—I: Imaginary points and lengths on real straight lines, Imaginary straight lines, Properties of semi-real figures, 1–40; II: The circle with a real branch, The conic with a real branch, 41–69; III: Angles between imaginary straight lines, Measurement of imaginary angles and of lengths on imaginary straight lines, Theorems connected with projection, 70–124; IV: The general conic, 125–140; V: The imaginary conic, 141–163; VI: Tracing of conics and straight lines, 164–195; VII: The imaginary in space, 196–212; Index of Theorems, 213–215; Index of terms and definitions, 216.

Aeronautics, A Class Text. By EDWIN BIDWELL WILSON. New York, Wiley, 1920. 8vo. 8 + 265 pages. Price $4.00.

Preface: "For several years I have been giving, at the Massachusetts Institute of Technology, courses of lectures on those portions of dynamics, both rigid and fluid, which are fundamental in aeronautical engineering. The more elementary parts of these courses, covering about ninety out of one hundred fifty lectures, are found in this book. Although it has been customary to teach the two subjects of rigid and of fluid dynamics in parallel or in rapid alternation, so that they are both developed as needed for each other and for the accompanying courses on airplane and airship design, it has seemed better in making a presentation in book form to separate them. The student should have completed Chaps. IX–XII of the fluid mechanics before undertaking the latter part of Chap. VI.

"A number of topics which might well be included in a work on aeronautics have been omitted from the book, as they are from my lectures, because they can be taken up so much better in the parallel courses on design. In the preparation of the selected material I have had constantly in mind my own experience and needs relative to effective classroom instruction, particularly in the matter of lists of exercises. Although my students are supposed to have completed thorough courses in calculus, including the elements of differential equations, and in theoretical and applied mechanics, it has seemed better to assume too little, rather than too much, as retained in usable form. I hope, therefore, that with the present interest in aeronautics in particular, and in applied mathematics in general, this work may prove stimulating to other than technical students of aeronautical engineering.

"Nobody can issue a book on aeronautics at this time without lamenting the fact that much, if not most, of the progress in theory which has been made during the war, particularly in England, has not yet been released for publication. To wait, however, until its release and subsequent digestion would mean a long delay. Indeed from one viewpoint no time is more appropriate for the printing of these elementary, introductory, and orienting lectures than just now when there impends a deluge of material for advanced study . . ."

Contents: *Introduction*—I. Mathematical preliminaries; 3–10; II. The pressure on a plane, 11–21; III. The skeleton airplane, 22–36. *Rigid Mechanics*—IV. Motion in a resisting medium, 37–56; V. Harmonic motion, 57–80; VI. Motion in two dimensions, 81–106; VII. Motion in three dimensions, 107–121; VIII. Stability of the airplane, 122–151. *Fluid Mechanics* —IX. Motion along a tube, 152–164; X. Planar motion, 165–181; XI. Theory of dimensions, 182–198; XII. Forces on an airplane, 199–217; XIII. Stream function, velocity potential, 218-235; XIV. Motion of a body in a liquid, 236–250; XV. Motion in three dimensions, 251–262. *Index*—263–265.

How to make and use Graphic Charts. By A. C. HASKELL, with an introduction by R. T. Dana. New York, Codex Book Co., 1919. 8vo. 7 + 539 pp.

Contents: Introduction, 1–7; Rectilinear charts, 8–11; Logarithmic charts, 12–16; Semi-logarithmic charts, 17–24; Polar charts, 25–26; Geometric charts, 27–29; Trilinear charts, 30–36; Nomographic or alignment charts, 37–53; General principles pertaining to the use of charts, 54–77; Organization and management charts, 78–167; Cast and cast analysis charts, 168–228; Scheduling and progress charts, 229–261; Operating characteristics, 262–308; Charts showing the results of tests and experiments, 309–325; Trends, tendencies and statistical prediction shown by charts, 326–341; Computation, arithmetical and geometrical, by charts, 342–444; Charts as an aid to designing and estimating, 445–508; Miscellaneous uses of charts, 509–534; Index, 535–539.

There are numerous exact references to the literature of the subject on pages 165–167, 227–228, 260–261, 307–308, 324–325, 340–341, 442–444, 506–508, 533–534.

NOTES.

An author and subject index (4 + 152 pages) for the first fifty years of the *Sitzungsberichte der mathematisch-physikalischen Klasse der kgl. Bayerischen Akademie der Wissenschaften zu München* was prepared by A. Hilsenbeck and published in 1913.

D. M. Y. Sommerville's *The Elements of Non-Euclidean Geometry* has appeared with "Chicago . . . Open Court . . . , 1919" on the title page. The work does not differ, except in title page, from that published by Bell of London, in 1914.

Rand McNally of Chicago published in 1918 *Plane Geometry* (Price $1.25) by MABEL SYKES and C. E. COMSTOCK. Mr. Comstock is a member of the Mathematical Association of America and a professor of mathematics in the Bradley Polytechnic Institute.

Two other members of the Association, MATILDA AUERBACH, supervisor of mathematics in the Ethical Culture High School, New York City, and C. B. WALSH, principal of the Friends Central School, Philadelphia, are the authors of *Plane Geometry* (Philadelphia, Lippincott, 1920, 18 + 383 pp. price $1.32), in the Lippincott's School Text Series edited by W. F. Russell, dean of the College of Education, State University of Iowa. The three volumes of *Applied Arithmetic, the Three Essentials* in this same series are by Professor N. J. LENNES, of the University of Montana, and Professor FRANCES JENKINS, of the University of Cincinnati. 1919–1920. 12mo. 11 + 283 pp.; 9 + 294 pp.; 9 + 340 pp. Price 72 + 80 + 88 cents).

Quotation from *The Degradation of the Democratic Dogma*. By Henry Adams. New York, Macmillan, 1919; "The Rule of Phase applied to history" (1909) pages 272–274:

". . . Static electricity already lay beyond the legitimate domain of sensual science, while beyond static electricity lay a vast supersensual ocean roughly called the ether, which the physicists and chemists, on their old principles, were debarred from entering at all, and had to be dragged into, by Faraday and his school. Beyond the ether, again, lay a vast region, known to them as the only substance which they knew or could know—their own thought,—which they positively refused to touch.

"Yet the physicists here, too, were helpless to escape the step, for where they refused to go as experimenters, they had to go as mathematicians. Without the higher mathematics they could no longer move, but with the higher mathematics, metaphysics began. There the restraints of physics did not exist. In the mathematical order, infinity became the invariable field of action, and not only did the mathematician deal habitually and directly with all sorts of infinities, but he also built up hyper-infinities, if he liked, or hyper-spaces, or infinite hierarchies of hyper-space. The true mathematician drew breath only in the hyper-spaces of Thought; he could exist only by assuming that all phases of material motion merged in the last conceivable share of immaterial motion—pure mathematical thought.

The physicist, in self-defence, though he may not deny, prefers to ignore this rigorous con-

sequence of his own principles, as he refused for many years to admit the consequences of Faraday's experiments; but at least he can surely rely upon this admission being the last he will ever be called upon to make. No phase of hyper-substance more subtle than thought can ever be conceived, since it could exist only as his own thought returning into itself. Possibly, in the inconceivable domains of abstraction, the ultimate substance may show other sides or extensions, but to man it can be known only as hyperthought,—the region of pure mathematics and metaphysics,—the last and universal solvent.

"There even mathematics must stop. Motion itself, ended; even thought became merely potential in this final solution. The hierarchy of phases was complete."

ARTICLES IN CURRENT PERIODICALS.

ANNALS OF MATHEMATICS, second series, volume 21, no. 2, December, 1919 (published February, 1920): "A property of cyclotomic integers and its relation to Fermat's last theorem" by H. S. Vandiver, 73–80; "Surfaces of rotation in a space of four dimensions" by C. L. E. Moore, 81–93; "The circle nearest to n given points, and the point nearest to n given circles" by J. L. Coolidge, 94–97; "Singular solutions of differential equations of the second order" by E. M. Coon, 98–103; "Note on a class of integral equations of the second kind" by C. E. Love, 104–117; "Concerning sense on closed curves in non-metrical plane analysis situs" by J. R. Kline, 118–119; "On the theory of summability" by G. James, 120–127; "On the consistency and equivalence of certain generalized definitions of the limit of a function of a continuous variable" by L. L. Silverman, 128–140.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 26, no. 4, January, 1920: "The October meeting of the American Mathematical Society" by F. N. Cole, 145–151; "The October meeting of the San Francisco Section" by B. A. Bernstein, 152–155; "On the proof of Cauchy's integral formula by means of Green's formula" by J. L. Walsh, 155–157; "A set of completely independent postulates for the linear order η" by M. G. Gaba, 158–159; "Certain properties of binomial coefficients" by W. D. Cairns, 160–164; "The work of Poincaré on automorphic functions" [review of *Oeuvres de Henri Poincaré*, tome II (Paris, 1916)] by G. D. Birkhoff, 164–172; "A brief account of the life and work of the late Professor Ulisse Dini" by W. B. Ford, 173–177; "Shorter Notices," 177–183; "Notes," 184–188; "New publications," 189–192—No. 5, February: "Integro-differential equations with constant limits of integration" by I. A. Barnett, 193–203; "On a pencil of nodal cubics" by N. Altshiller-Court, 203–211; "Definition and illustrations of new arithmetical group invariants" by E. T. Bell, 211–223; "Matrices and determinoids" [review of C. E. Cullis's *Matrices and Determinoids* (Cambridge, 1913–1918)] by J. B. Shaw, 224–233; "Notes," 233–237; "New publications," 237–240.

L'ENSEIGNEMENT MATHÉMATIQUE, volume 20, no. 6, January, 1920: "Sur l'élimination algébrique" by C. Riquier, 405–421; "Nouveaux théorèmes sur le viriel de forces et leurs applications géométriques et mécaniques" by F. Boulad, 421–432; "Sur les foyers rationnels d'une courbe algébrique" by E. Turrière, 433–436; Chronique, Bibliographie, Bulletin Bibliographique, 436–462; Table de Matières, 463–468; Supplément, "Spaco" by R. de Saussure, 12 pages [Specimen of the international language "Espérantide (conciliation de l'Esperanto et de l'Ido)." The first sentences are: "En spaco existas sep, kay nur sep, figuron, kiun estas nur pozician, t.e., kiun entenas nenia grando. Lu estas (fig. 1): 1. La *punkto* (*P*). Irg ni punkto estas nura pozicio kay havas neni amplexo; punkto povas rotaci omnimanere sur si self, ne cestante esti la sama punkto.

2. La *reglo* (*R*), or rekta senfina linio, konsiderata kom spacelemento, kom nedivizibla tuto (kay ne kom serio de punkton). Irg ni reglo estas nura pozicio kay enhavas nenia grando; reglo povas gliti or rotaci sur si self, ne cestante esti la sama reglo."].

THE MATHEMATICS TEACHER, volume 12, no. 2, December, 1919: "Certain undefined elements and tacit assumptions in the first book of Euclid's Elements" by H. E. Webb, 41–60; "Association of mathematics teachers of New Jersey. Report of the committee of first-year high-school mathematics," 61–74; "New books," 75–76; "Notes and News," 77–88.

MONIST, volume 30, no. 1, January, 1920: "The analytical treatment of Newton's problems" by P. E. B. Jourdain, 19–36 [First paragraph: "It is a great help to the proper understanding of the meaning of Newton's propositions to consider the formulation of these propositions in analytical language. In this article, after a mention is made of Leibniz's analytical work of 1689 (§ I), Varignon's of 1690 (§ II), Maclaurin's of 1742 (§ III), and of some work of Johann Bernoulli, Keill, Hermann, and others near the beginning of the eighteenth century (§ IV), the contributions

of Lagrange (1788) and the integrals of the dynamical equations are considered in greater detail (§§ V, VI). In connection with these integrals, the problems of the first three and then two bodies are treated in the most general way possible (§ VII), and, in § VIII, a single case of integration noticed by Jacobi is given and some further simplifications added. In § IX a short analytical summary of much of Newton's work on the dynamics of a particle is given. In § X the work of Möbius, Hamilton, and Grassmann on vector methods, which is of such great importance to analytical mechanics, is indicated, and the logical problem of the principles of mechanics and the light thrown on it by this work are discussed"]; "Space and the world in space" by J. Dieserud, 37–56; "The new number of the cosmic sand. On the extent of the universe and the limits of space and time" by H. Friedenthal, 57–69.

Nouvelles Annales de Mathématiques, volume 78, November, 1919: "Cyclides du quatrième degré" by R. Dontot, 401–417; "Théorème sur les courbes planes" by R. Bricard, 418–421; "Triangles et quadrilatères de Poncelet" by G. Fontené, 421–424; "Distance du centre de la sphère circonscrite, au centre de gravité du tétraèdre" by V. Thebault, 424–426; "Sur l'aire d'un polygone" by V. Jamet, 426–435; Concours spécial d'admission à l'Ecole Polytechnique en 1919. Sujets de composition, 435–437; solutions de questions proposées, 437–440.

Proceedings of the National Academy of Science of the U. S. of A., volume 5, no. 12, December, 1919: "Conditions necessary and sufficient for the existence of a Stieltjes integral" by R. D. Carmichael, 551–555; "Transformations of cyclic systems of circles" by L. P. Eisenhart, 555–757—Volume 6, no. 1, January, 1920: "The commutativity of one-parameter transformations in real variables" by A. C. Lunn, 24–26.

Quarterly Journal of Pure and Applied Mathematics, volume 48, no. 4, January, 1920: "A new solution of Waring's problem" (continued) by G. H. Hardy and J. E. Littlewood, 289–293; "A class of definite integrals" by S. Ramanujan, 294–310; "Motion of one solid on another" by A. B. Basset, 310–320; "Generalisation of a Theorem due to Sonine" by J. W. Nicholson, 321–329; "The value of the definite integral

$$\int_{-\infty}^{\infty} \frac{e^{at^2+bt}dt}{e^{ct}+d}"$$

by L. J. Mordell, 329–342; "On sequences of integers defined by recurrence relations" by R. D. Carmichael, 343–372; "On inverses of ∇^2 and other quadratic operators" by E. B. Elliott, 372–379; "Note on compound determinants expressible as simple determinants," T. Muir, 379–384.

Revista Matemática Hispano-Americana, volume 1, no. 10, December, 1919:"Pedro Sanchez Ciruelo" by J. M. Lorente, 301–304; "Matemática de precesión y matemática de aproximación" by F. Klein, 305–314 [Translation of a lesson in his *Anwendung der Differential-und Integralrechnung auf Geometrie*, Leipzig, 1907]; "La sucesión de Fibonacci" (fourth article) by F.Vera, 315–322; "Restitución de una de las obras perdidas de Euclides" by V. R. y. Prósper, 323-325 [Refers among other works to R. C. Archibald's *Euclid's Book on Division of Figures* (1915)]; "Dos obras recientes sobre historia de la matemática" by G. A. Miller, 325–326 [Review of L. E. Dickson's *History of the Theory of Numbers*, volume 1 (1919) and F. Cajori's *History of Mathematics*, Revised edition (1919).

Revue de l'Enseignement des Sciences, volume 13, nos. 125–126, May-June, 1919: "Sur l'égalité et la similitude des figures en géométrie plane" by J. Lemaire, 129–143; "A propos d'enveloppes" by F. Meyer, 143–145; "Premières leçons de géométrie" by J. Juhel-Rénoy, 146–154; "Sur les trièdres" by J. Juhel-Rénoy, 154–155; "Paul Chauvet" (1858–1918) by E. Brucker, 156–157; "Examens et concours de 1918 et de 1919," 158–173; "Bibliographie," 173–176.

Revue Scientifique, volume 58, no. 2, January 24, 1920: "Carlo Bourlet, sa vie et son oeuvre (1866–1913)" by A. Boulanger, 42–45.

School and Society, volume 11, February 21, 1920: "Elementary statistics in high school mathematics as a socializing agency" by T. L. Kelley, 228–230.

Scientific Monthly, volume 10, no. 4, April, 1920: "The measure of excellence in scientific activity" by R. D. Carmichael, 343–359.

Transactions of the Academy of Science of St. Louis, volume 23, no. 9, March 2, 1920: "A study of the properties of integral numbers" and "New evidence of a relation between gravitation and electrical action, and of local changes in the electrical potential of the earth" by F. E. Nipher, 373–387.

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F. R. MORRIS, "Classification of involutory cubic space transformations," *University of California Publications in Mathematics*, vol. 1, no. 11, February, 1920, pp. 223–240. (University of California, 1918).

A. R. WILLIAMS, "On a birational transformation connected with a pencil of cubics," *University of California Publications in Mathematics*, vol. 1, no. 10, February, 1920, pp. 211–222. (University of California, 1916).

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

[Send all communications about problems and solutions to B. F. Finkel, Springfield, Mo.]

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them*. If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2834. Proposed by OTTO DUNKEL, Washington University.

In any triangle ABC let M and N be, respectively, the points in which the median and the bisector of the angle at A meet the side BC, Q and P the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced. Prove that the straight line QO is perpendicular to BC and the similar theorem for the external bisector of the angle at A.

This proposition shows the relation between two constructions for the center of curvature O of a conic for which B and C are foci and A is a point of the conic. (The figure also gives an easy proof of the Law of Tangents for triangles compare *1920*, 53–54. See also *1920*, 226.)

2835. Proposed by J. L. RILEY, Stephenville, Texas.

If x, y, z, u are finite, and not all zero, and satisfy the equations

$$x = by + cz + du, \qquad y = ax + cz + du, \qquad z = ax + by + du, \qquad u = ax + by + cz,$$

and if none of the quantities a, b, c, d, have the value -1, then will

$$\frac{a}{1+a} + \frac{b}{1+b} + \frac{c}{1+c} + \frac{d}{1+d} = 1.$$

2836. Proposed by W. V. N. GARRETSON, Rutgers College.

A ladder 40 feet long rests with one end on the ground against the foot of a building and the other end against the side of a second building directly across the street from the first. A second ladder 25 feet long inclines in a similar manner from the foot of the second building against the side of the first building, the two ladders crossing at a point 15 feet above the ground. How wide is the street?

2837. Proposed by B. F. FINKEL, Drury College.

Assuming v to be the velocity of sound; $\pm u_s$ the velocity of S, the source of sound, and n its frequency; $\pm u_r$ the velocity of R, the receiver; and $\pm w$ the velocity of M, the medium, discuss fully Döppler's Principle for the apparent frequency n'. The double signs are used to indicate that the discussion is to include the cases when source and receiver are approaching and when separating and the same consideration with reference to the medium. Limiting cases are especially desirable; e.g., when $w = 0$, $u_s = 0$, and $-u_r = 2v$; $w = 0$, $+u_s = v$, and $u_r = 0$; etc.

2838. "A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on to the other end. Suppose that the monkey begins to climb the rope, what will be the result?"

This problem was invented by Lewis Carroll in December, 1893 (S. D. Collingwood, *The Life and Letters of Lewis Carroll (Rev. C. L. Dodgson)*, New York, 1899, pp. 317–318) and in his diary he remarked: "Got Professor Clifton's answer [R. B. Clifton, professor of physics at Oxford] to the 'Monkey and Weight Problem.' It is very curious, the different views taken by good mathematicians. Price [Bartholomew Price, professor of physics at Oxford] says that the weight goes *up* with increasing velocity; Clifton (and Harcourt [A. G. Vernon-Harcourt, professor of chemistry at Oxford]) that it goes *up*, at the same rate as the monkey; while Sampson [probably E. F. Sampson, lecturer, tutor and censor of Christ Church, Oxford] says that it goes *down*." Yet another solution by Rev. A. Brook is given on page 268 of *The Lewis Carroll Picture Book* . . . edited by S. D. Collingwood (London, 1899), namely that "the weight remains stationary."

The problem has been recently discussed in *School Science and Mathematics*, volume 17, December, 1917, p. 821; volume 19, December, 1919, p. 815; and volume 20, February, 1920, pp. 172–173. The editors of the MONTHLY invite mathematical solutions of the problem.

2839. By translating the steps of the construction of a regular pentagon from plane geometry into algebra show that one of the fifth roots of unity is equal to

$$\tfrac{1}{4}(\sqrt{5}-1)+\frac{i}{4}\sqrt{10+2\sqrt{5}}.$$

(This problem is proposed for solution in Wilczynski and Slaught, *College Algebra with Applications*, Boston, 1916, p. 193.)

2840. Proposed by NORMAN ANNING, University of Maine.

It is observed in a table of values of

$$\log_{10}(\operatorname{colog}_{10} x)$$

that second differences are zero for values of x in the neighborhood of 0.37. Prove that this must be the case. (Cf. Chappell's *Five-Figure Mathematical Tables*, Edinburgh, 1915, p. 180.)

2841. Proposed by WILLIAM HOOVER, Columbus, Ohio.

The mixed number,

$$9\frac{49}{64}=9+\frac{49}{64}=3^2+\frac{7^2}{8^2},$$

is of the type form

$$k^2+\frac{(2k+1)^2}{(2k+2)^2};$$

how may the *forms* of the terms of the fractional part be determined *deductively?*

Generally required that

$$k^2+\frac{\{\varphi_1(k)\}^2}{\{\varphi_2(k)\}^2}$$

be a perfect square; show how $\varphi_1(k)$ and $\varphi_2(k)$ may be found.

2842. Proposed by H. S. UHLER, Yale University.

Express explicitly the following sextic in x as the product of a quadratic and a biquadratic:

$$3x^6-6k_1x^5+(7k_1^2-9k_2^2)x^4-2(2k_1^3-4k_1k_2^2-3k_3^2)x^3+[(k_1^2-k_2^2)^2-9k_1k_3^3]x^2$$
$$-(k_1^2-2k_2^2)(k_1k_2^2-9k_3^3)x+(k_1^2-3k_2^2)(k_2^4-3k_1k_3^3).$$

SOLUTIONS OF PROBLEMS.

2746 [1919, 72]. Proposed by S. A. COREY, Des Moines, Iowa.

Establish the following algebraic identity without actually performing the indicated operations:

$$2(t_1t_2+c_1t_3t_4+c_2t_5t_6+c_1c_2t_7t_8)(r_1r_2+c_1r_3r_4+c_2r_5r_6+c_1c_2r_7r_8)$$
$$=(r_1t_1-c_1r_3t_3-c_2r_5t_5+c_1c_2r_7t_7)(r_2t_2-c_1r_4t_4-c_2r_6t_6+c_1c_2r_8t_8)$$
$$+(r_1t_2-c_1r_3t_4-c_2r_5t_6+c_1c_2r_7t_8)(r_2t_1-c_1r_4t_3-c_2r_6t_5+c_1c_2r_8t_7)$$
$$+c_1(r_1t_3+r_3t_1-c_2r_5t_7-c_2r_7t_5)(r_2t_4+r_4t_2-c_2r_6t_8-c_2r_8t_6)$$

$$+ c_1(r_1t_4 + r_2t_2 - c_2r_3t_5 - c_2r_7t_6)(r_2t_3 + r_4t_1 - c_2r_6t_7 - c_2r_8t_5)$$
$$+ c_2(r_1t_5 + c_1r_3t_7 + r_3t_1 + c_1r_7t_3)(r_2t_6 + c_1r_4t_8 + r_6t_2 + c_1r_8t_4)$$
$$+ c_2(r_1t_6 + c_1r_3t_8 + r_4t_2 + c_1r_7t_4)(r_2t_5 + c_1r_4t_7 + r_6t_1 + c_1r_8t_3)$$
$$+ c_1c_2(r_1t_7 - r_3t_5 + r_3t_3 - r_7t_1)(r_2t_8 - r_4t_6 + r_6t_4 - r_8t_2)$$
$$+ c_1c_2(r_1t_8 - r_3t_6 + r_4t_4 - r_7t_2)(r_2t_7 - r_4t_5 + r_6t_3 - r_8t_1).$$

By assuming special relations between the constants involved show that the product of the sum of four squares by the sum of four squares equals the sum of four squares.

Solution by the Proposer.

We have the well-known identity:

$$(T_1^2 + c_1T_2^2 + c_2T_3^2 + c_1c_2T_4^2)(R_1^2 + c_1R_2^2 + c_2R_3^2 + c_1c_2R_4^2) = P_1^2 + c_1P_2^2 + c_2P_3^2 + c_1c_2P_4^2, \quad (1)$$

where

$$P_1 = T_1R_1 - c_1T_2R_2 - c_2T_3R_3 + c_1c_2T_4R_4, \qquad P_2 = T_2R_1 + T_1R_2 - c_2T_3R_4 - c_2T_4R_3,$$
$$P_3 = T_3R_1 + T_1R_3 + c_1T_4R_2 + c_1T_2R_4, \quad \text{and} \quad P_4 = T_4R_1 - T_1R_4 - T_3R_2 + T_2R_3.$$

Assuming R_n to be a vector (or even a quaternion), (1) will hold as to its scalar part, being a homogeneous quadratic.

Let $\theta = \sqrt{-1}$, $H = \frac{1}{2}[1 - j - \theta(i + k)]$, $I = \frac{1}{2}[1 + j + \theta(i - k)]$, $J = \frac{1}{2}[1 + j - \theta(i - k)]$, and $K = \frac{1}{2}[1 - j + \theta(i + k)]$, i, j, and k being Hamilton's unit vectors. Also let

$$R_1 = r_1H + s_1I + s_2J + r_2K, \qquad R_2 = r_3H + s_3I + s_4J + r_4K,$$
$$R_3 = r_5H + s_5I + s_6J + r_6K, \quad \text{and} \quad R_4 = r_7H + s_7I + s_8J + r_8K.$$

If $r_{2n-1} + s_{2n-1} + s_{2n} + r_{2n} = 0$, R_n becomes a vector but its tensor is usually imaginary, its square being $2(r_{2n-1}r_{2n} + s_{2n-1}s_{2n})$ (v. Vol. XIV, pp. 19–22 of Monthly). If $s_{2n} = 0$, this reduces to $2(r_{2n-1}r_{2n})$, and if we further assume that $s_{2n-1} = -(r_{2n-1}r_{2n})$, R_n becomes a vector the squared tensor (or norm) of which is $2(r_{2n-1}r_{2n})$. Substituting these vector values of R_1, R_2, R_3, and R_4 in P_1, P_2, P_3, and P_4, the norm of P_n is found to be $2p_{2n-1}p_{2n}$, where

$$p_1 = T_1r_1 - c_1T_2r_3 - c_2T_3r_5 + c_1c_2T_4r_7, \qquad p_2 = T_1r_2 - c_1T_2r_4 - c_2T_3r_6 + c_1c_2T_4r_8,$$
$$p_3 = T_2r_1 + T_1r_3 - c_2T_4r_5 - c_2T_3r_7, \qquad p_4 = T_2r_2 + T_1r_4 - c_2T_4r_6 - c_2T_3r_8,$$
$$p_5 = T_3r_1 + c_1T_4r_3 + T_1r_5 + c_1T_2r_7, \qquad p_6 = T_3r_2 + c_1T_4r_4 + T_1r_6 + c_1T_2r_8,$$
$$p_7 = T_4r_1 - T_3r_3 + T_2r_5 - T_1r_7, \quad \text{and} \quad p_8 = T_4r_2 - T_3r_4 + T_2r_6 - T_1r_8.$$

Substituting in (1) and reducing, we get

$$(T_1^2 + c_1T_2^2 + c_2T_3^2 + c_1c_2T_4^2)(r_1r_2 + c_1r_3r_4 + c_2r_5r_6 + c_1c_2r_7r_8) = p_1p_2 + c_1p_3p_4 + c_2p_5p_6 + c_1c_2p_7p_8. \quad (2)$$

Similarly let $T_n = t_{2n-1}H + u_{2n-1}I + u_{2n}J + t_{2n}K$, where $u_{2n-1} = 0$ and $u_{2n} = -(t_{2n-1} + t_{2n})$. T_n will then be a vector.

Substituting these values of T_n in (2) and taking scalars the given identity is obtained. This process of taking scalars is simplified if we observe that if $Q_1 = x_1H + y_1I + z_1J + v_1K$, and $Q_2 = x_2H + y_2I + z_2J + v_2K$ are vectors, and if $y_1 = y_2 = 0$, $S{\cdot}Q_1Q_2 = -(x_1v_2 + v_1x_2)$.

If $c_1 = c_2 = 1$, $t_{2n-1} = t_{2n}$, and $r_{2n-1} = r_{2n}$ in the given identity, the product of the sum of four squares by the sum of four squares is seen to be the sum of four squares.

Further if t_{2n-1} be the conjugate of t_{2n}, and r_{2n-1} be the conjugate of r_{2n}, and if $c_1 = c_2 = 1$, in the given identity, it becomes 2 (sum of eight squares) (sum of eight squares) = (sum of 16 squares). This is a special case of

$$\left(\sum_{n=1}^{16} x_n^2\right)\left(\sum_{n=1}^{16} y_n^2\right) = \sum_{n=1}^{16} z_n^2,$$

and establishes the theorem:

The product of the sum of sixteen squares of numbers forming the legs of eight Pythagorean triangles by the sum of sixteen squares, eight of which have the same sum as the remaining eight, equals the sum of sixteen squares.

Proof of the given identity could have been obtained by the use of the ordinary complex quantities of algebra, but the above proof is given because of its greater generality and novelty.

It will be observed that the given identity as well as (1), (2), and the above 16-square theorem, all being homogeneous algebraic quadratic identities, may be given geometric interpretations by letting certain of the letters represent vectors and then taking the scalars of the resulting expressions.

2748 [1919, 72]. Proposed by J. B. REYNOLDS, Lehigh University.

The vertices of a triangle are (0, 0), $(2a, 0)$, and $(2x, 2y)$. Where are the vertices of the triangle of least area having its vertices on the perpendicular bisectors of the sides of the given triangle and the same center of gravity as the given triangle?

SOLUTION BY A. M. HARDING, University of Arkansas.

Let $Q_1(x_1, y_1)$, $Q_2(x_2, y_2)$, $Q_3(x_3, y_3)$ be the vertices of the required triangle. Since Q_1, Q_2, Q_3, are on the perpendicular bisectors of the sides of the triangle $P_1P_2P_3$, we have

$$y_1y + x_1(x - a) - x^2 + a^2 - y^2 = 0, \tag{1}$$

$$y_2y + x_2x - x^2 - y^2 = 0, \tag{2}$$

$$x_3 = a. \tag{3}$$

If the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity

$$\frac{x_1 + x_2 + x_3}{3} = \frac{2x + 2a}{3}, \qquad \frac{y_1 + y_2 + y_3}{3} = \frac{2y}{3},$$

or

$$x_1 + x_2 = 2x + a, \tag{4}$$

$$y_1 + y_2 + y_3 = 2y. \tag{5}$$

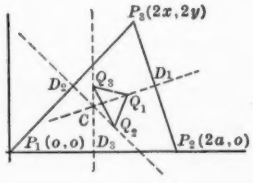
From equations (1), (2), (4), (5), we find

$$\begin{aligned} ax_1 &= -yy_3 + ax + a^2, \\ ax_2 &= yy_3 + ax, \\ ay_1 &= (x - a)y_3 + ay, \\ ay_2 &= -xy_3 + ay. \end{aligned} \tag{6}$$

The area of triangle $Q_1Q_2Q_3$ is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \frac{1}{2a^2}\begin{vmatrix} ax_1 & ay_1 & 1 \\ ax_2 & ay_2 & 1 \\ ax_3 & ay_3 & 1 \end{vmatrix}$$

$$= \frac{1}{2a^2}\begin{vmatrix} -yy_3 + ax + a^2, & (x - a)y_3 + ay, & 1 \\ yy_3 + ax, & -xy_3 + ay, & 1 \\ a^2, & ay_3, & 1 \end{vmatrix}$$



whence $2a\Delta = 3yy_3^2 - 2(x^2 + y^2 - ax + a^2)y_3 + a^2y$.

The area will be a minimum when $(d/dy_3)(2a\Delta) = 0$; that is, when

$$3yy_3 = x^2 + y^2 - ax + a^2.$$

It may be easily shown that the center of the circumcircle of $\Delta P_1P_2P_3$ is $C(a, y_0)$, where $yy_0 = x^2 + y^2 - ax$. Hence $3yy_3 = yy_0 + a^2$, or $y_3 = y_0/3 + a^2/3y$. The coördinates of the other vertices may now be found from equations (6).

Note. It may be shown that if the triangles $P_1P_2P_3$ and $Q_1Q_2Q_3$ have the same center of gravity,

$$\frac{Q_1D_1}{P_2P_3} = \frac{Q_2D_2}{P_3P_1} = \frac{Q_3D_3}{P_1P_2},$$

where D_1, D_2, D_3 are the mid-points of the sides of $\Delta\, P_1P_2P_3$. This property of the triangles might have been used in this problem.

Also solved by E. H. CLARKE, C. E. HORNE, and A. PELLETIER.

2749 [1919, 72]. Proposed by C. N. SCHMALL, New York City.

In the parabola, $y^2 = 4ax$, two normals to the curve are drawn at the ends of a focal chord. Show that the area between these normals and the curve is $20a^2/(3 \sin^3 2\phi)$ where ϕ is the angle between one of the normals and the x-axis.

SOLUTION BY H. M. ROESER, Bureau of Standards, Washington, D. C.

The tangents to a parabola at the extremities of a focal chord intersect on the directrix at right angles. (Tanner and Allen, *Analytic Geometry*, page 227.) The tangents and normals will, therefore, form a rectangle of which the focal chord is a diagonal and whose area is equal to the product of the lengths of the tangents from their intersection on the directrix to the points of tangency. The area sought is the area of one of the triangular halves of the rectangle plus two-thirds of the area of the other triangle or five-sixths of the area of the rectangle.

Let m = slope of one of the normals. Then $y = mx - 2am - am^3$ is the equation of one normal and $y = -x/m + 2a/m + a/m_0$ is the equation of the other normal. $y = -x/m - am$ is the equation of one tangent, and $y = mx + a/m$ is the equation of the other tangent. The tangents intersect at the point $(x, y) \equiv [-a, a(1 - m^2)/m]$ and touch the curve at $(x, y) \equiv [am^2, -2am]$ and $(x, y) \equiv [a/m^2, 2a/m]$, respectively.

The lengths of the tangents are $l_1 = a(1 + m^2)\sqrt{1 + m^2}/m$ and $l_2 = a(1 + m^2)\sqrt{1 + m^2}/m^2$. The area sought is therefore $5l_1l_2/6 = 5a^2(1 + m^2)^3/6m^3 = 5a^2/(6 \sin^3 \phi \cos^3 \phi) = 20a^2/(3 \sin^3 2\phi)$.

Also solved by E. H. CLARKE, H. H. DOWNING, POLYCARP HANSEN, C. E. HORNE, MARCIA L. LATHAM, A. PELLETIER, and the PROPOSER.

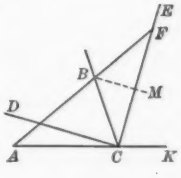
2750 [1919, 72]. Proposed by A. CAMPBELL, St. Johnsburg, Vermont.

Given the base, the sum of the sides of the triangle and the difference of the base angles, to construct the triangle.

SOLUTION BY THE PROPOSER.

Let b, be the given base; $a + c$, the sum of the other two sides, and $\alpha = C - A$ the difference of the base angles.

On the line AK lay off $AC = b$ and at C construct an angle $ACD = \frac{1}{2}(C - A) = \frac{1}{2}\alpha$. Draw CE perpendicular to DC. With A as a center and a radius equal to $a + c$ describe an arc intersecting CE in F. Draw AF. Construct the angle BCF equal to the angle BFC. Then the triangle ABC is the required triangle.



For, triangle BCF is an isosceles triangle having its base angles equal by construction. Hence, $BC = BF$, and, therefore, $AB + BC = AF$.

Also, angle CBF = angle A + angle C, or angle MBC (BM being the bisector of angle CBF) = $\frac{1}{2}$ angle CBF = $\frac{1}{2}$(angle A + angle C) = angle BCD = angle BDC = angle A + angle DCA; whence angle DCA = $\frac{1}{2}$(angle C − angle A) = $\frac{1}{2}\alpha$.

Also solved by C. L. ARNOLD, GEORGE AQUIS, MARY BEJSORIC, P. J. DA CUNHA, CHANG CHIH-CHEN, H. H. DOWNING, A. M. HARDING, C. E. HORNE, MARCIA L. LATHAM, A. PELLETIER, MARIAN M. TORREY, and LOUIS WEISNER.

2751 [1919, 72]. Proposed by ENOS E. WITMER, Senior in Franklin and Marshall College.

Investigate the problem of solving the equation

$$x^4 + ay^4 = w^2 + av^2. \tag{1}$$

Carmichael's *Diophantine Analysis*, Problem 18, p. 54.

SOLUTION BY THE PROPOSER.

It appears that x, y, v, w, and a are to be rational numbers and $a \neq 0$.
If $v^2 = y^4$, $w^2 = x^4$, and a solution is given by $v = \pm y^2$ $w = \pm x^2$.
If $v^2 \neq y^4$ we may proceed as follows: From the given equation, we have

$$x^4 - w^2 = a(v^2 - y^4); \tag{2}$$

whence,

$$\frac{x^4 - w^2}{v^2 - y^4} = a = \frac{(x^4 - w^2)(v^2 - y^4)}{(v^2 - y^4)^2}, \tag{3}$$

or

$$\left(\frac{x^2v \pm wy^2}{v^2 - y^4}\right)^2 - \left(\frac{x^2y^2 \pm wv}{v^2 - y^4}\right)^2 = a. \tag{4}$$

Letting

$$\frac{x^2v \pm wy^2}{v^2 - y^4} = b, \tag{5}$$

$$\frac{x^2y^2 \pm wv}{v^2 - y^4} = c \tag{6}$$

and solving (5) and (6) for x^2 and w,

$$x^2 = bv - cy^2 \tag{7}$$

and

$$w = cv - by^2. \tag{8}$$

From (7),

$$v = \frac{x^2 + cy^2}{b}, \tag{9}$$

Substituting the value of v from (9) in (8), we have

$$w = \frac{cx^2 + c^2y^2 - b^2y^2}{b}.$$

But from (4), (5), and (6), $b^2 - c^2 = a$. Hence if we put $b - c = m$, $b + c = a/m$; then

$$b = \frac{1}{2}\left(\frac{a}{m} + m\right), \qquad c = \frac{1}{2}\left(\frac{a}{m} - m\right).$$

Hence

$$x = r, \qquad y = s,$$

$$v = \frac{2r^2 + \left(\frac{a}{m} - m\right)s^2}{\frac{a}{m} + m} = \frac{2mr^2 + (a - m^2)s^2}{a + m^2},$$

$$w = \frac{\left(\frac{a}{m} - m\right)r^2 - 2as^2}{\frac{a}{m} + m} = \frac{(a - m^2)r^2 - 2ams^2}{a + m^2}.$$

2765 [1919, 171]. Proposed by A. M. HARDING, University of Arkansas.

ABC is an equilateral triangle. A point D is taken in BC such that BD is $\frac{1}{3}$ of BC and E is taken in CA such that CE is $\frac{1}{3}$ of CA. If the lines AD and BE intersect at O, show that OC is perpendicular to AD.

SOLUTION BY THE LATE L. G. WELD.

Since CD is twice CE and $\angle DCE = 60°$ the auxiliary line DE is perpendicular to CE; whence the point E is in the circumference of a circle described upon CD as a diameter. Since the triangles BCE and BOD are similar

$$BD \cdot BC = BO \cdot BE.$$

Hence, O, as well as E, lies in the circumference of the above circle and the angle COD, being inscribed in a semicircle, is a right angle.

Also solved by I. V. BACO, G. BREIT, H. H. DOWNING, WILLIAM HERBERG, I. HERMAN, J. A. MCWILLIAMS, C. E. MANGE, H. L. OLSON, A. PELLETIER, H. R. PHALEN, J. B. REYNOLDS, W. F. RIGGE, W. G. SIMON, and the PROPOSER.

NOTES AND NEWS.

EDITED BY E. J. MOULTON, Northwestern University, Evanston, Ill.

The announcement in this MONTHLY (*1920*, 88) that Assistant Professor FLORENCE P. LEWIS had been promoted to an associate professorship in mathematics was somewhat belated, as this promotion took place in 1913.

At Cornell University Mr. D. S. MORSE of Union College, Professor W. L. G. WILLIAMS, of William and Mary College, Mr. H. PORITSKY and Mr. H. M. LUFKIN (*1920*, 43) have been appointed instructors in mathematics for next year. Professor JAMES MCMAHON has been granted leave of absence for the year 1920–1921, Professor ARTHUR RANUM for the first term, and Professor F. R. SHARPE for the second term.

At Adelbert College of Western Reserve University, Dr. W. G. SIMON has been promoted to an assistant professorship, and Dr. C. A. NELSON of the University of Kansas has been appointed to an instructorship in mathematics.

Professor J. L. JONES, of Syracuse University, has been appointed head of the department of mathematics at Akron University.

At the College of the City of New York, Mr. J. A. BREWSTER has been promoted to an assistant professorship of mathematics, and Mr. W. A. WHYTE to an instructorship in mathematics.

Dr. MAELYNETTE ALDRICH, professor of mathematics, at Martha Washington College, a member of this Association, died from influenza on February 22.

Dr. O. A. RANDOLPH, associate professor of physics in the University of Colorado, a member of this Association, lost his life in a snow storm on April 11, during a mountain-climbing trip made for the purpose of photographing winter storm scenes.

Mr. J. W. LASLEY, Jr., assistant professor of mathematics at the University of North Carolina, has been on leave of absence and studying at the University of Chicago as fellow.

Dr. O. D. KELLOGG has been appointed associate professor of mathematics at Harvard University (correction of a previous statement, *1920*, 238).

Professor J. E. Rowe, of Pennsylvania State College, has resigned to accept an appointment as mathematics and dynamics expert in the ordnance bureau of the war department. He is serving as chief ballistician at the Aberdeen Proving Ground.

Dr. Gaston Julia (*1919*, 223) is maître de conférences de mathématiques at the Ecole des Hautes Etudes, Paris.

Dr. J. A. Brashear, manufacturer of astronomical and physical instruments at Pittsburgh, Pa., and member of many scientific societies, died on April 8, 1920 in his eightieth year.

Dr. Woldemar Voigt, professor of theoretical physics at the University of Göttingen since 1883, died December 13, 1919 at the age of sixty-nine years.

Vice-Admiral C. A. Campos Rodrigues, director of the obseravtory of Lisbon, died December 25, 1919 aged eighty-three years.

Professor Moritz Benedikt Cantor, who died at Heidelberg, April 10, 1920, was born at Mannheim, August 23, 1829. He received his doctorate sixty-nine years ago at the University of Heidelberg where he was appointed extraordinary professor in 1853 and honorary professor in 1877. Apart from his monumental *Vorlesungen über Geschichte der Mathematik*, 1880–1907, he was the author of: *Ueber ein wenig gebräuchliches Coordinatensystem* (diss.) 1851; *Grundzüge einer Elementar-Arithmetik*, 1855; *Mathematische Beiträge zum Kulturleben der Völker*, 1863; *Euklid und sein Jahrhundert*, 1867 (Italian translation, 1873); *Die römischen Agrimensoren und ihre Stellung in der Geschichte der Feldmesskunst, eine historisch-mathematische Untersuchung*, 1875; *Politische Arithmetik oder die Arithmetik des täglichen Lebens*, 1898; second edition, 1903.

Professor A. S. Eddington has been appointed President of the Section of Mathematics and Physics of the British Association for the Advancement of Science.

Professor Paul Appell, honorary dean of the Faculty of Sciences of the University of Paris has been appointed Rector of the Académie de Paris, in succession to the late Lucien Poincaré.

Dr. Emile Borel, recently professor of the theory of functions in the University of Paris, has, at his own request, been appointed professor of the calculus of probabilities and mathematical physics as successor to Professor Boussinesq, who has retired (*1919*, 419).

At the meeting of the American Mathematical Society in New York on April 24, the afternoon session was devoted to a symposium on relativity. Papers

were presented as follows: By Professor LEIGH PAGE, "The physical and philosophical significance of the principle of the theory of relativity and Einstein's theory of gravitation;" by Professor L. P. EISENHART, "Geometric aspects of the Einstein theory."

At the meetings of the Division of Physical Sciences of the National Research Council on April 28–29, 1920, the following motions with reference to mathematics were adopted:

1. That the MATHEMATICAL ASSOCIATION OF AMERICA be asked to nominate one member for the Division of Physcial Sciences of the National Research Council.

2. That the Division of Physical Sciences recommend to the National Research Council that the American Section of Mathematics, organized under the statutes adopted by the Division, be made the authorized agent of the Council in the negotiations leading to the organization of the proposed International Mathematical Union and become its representative in that international body when it shall have been organized. Further, the Division recommends that upon the formal organization of the International Mathematical Union the Council signify its adhesion and that the annual contribution be paid as soon as funds are available.

3. That the number of members at large be increased by one with the understanding that the first man elected to this membership should represent mathematics.

4. That the Division approve of the project for the publication of a Journal of Mathematical Abstracts and recommend the appointment of a committee to work out details and to take steps in consultation with the Executive Committee of the Division and the Finance Committee of the Council, looking toward the securing of the necessary funds.

5. That a committee be appointed to consider further and in detail the project for securing a revolving fund to assist in the publication of important scientific books, monographs and translations which are so unprofitable from a commercial standpoint as not to appeal to regular publishing houses.

6. That the Division approve the project for the establishment of one or more Research Fellowships in Mathematics, such Fellowships to be administered by the Division, and that the Division request the assistance of the Finance Committee of the Council in securing funds for this purpose.

The Division considered also the recommendations of its Committee on Coöperation in Mathematical Projects (consisting of the representatives of the American Mathematical Society in the Division [see *1920*, 239] and Professors E. R. HEDRICK, and R. C. ARCHIBALD) regarding committees on two special branches of mathematics offering advantages in coöperative research, and the formation of these committees was left to the chairman.

The American Section of the International Mathematical Union referred to above is, according to its statutes, composed of: the president and secretary of the American Mathematical Society and its three representatives on the National Research Council, four other members selected by the American Mathematical Society, three members selected by the Mathematical Association of America, and one member each by the National Academy of Sciences, the American Astronomical Society, the American Physical Society, and the American Association for the Advancement of Science.

Additional announcements of mathematics courses in Summer Sessions (cf. *1920*, 192–194; 235–236) are as follows:

University of California, June 21–July 31. By Professor D. N. LEHMER: Theory of numbers; Special advanced study and research. By Professor R. G. D.

RICHARDSON: Plane trigonometry; Theory of functions of a complex variable. By Professor G. E. F. SHERWOOD: Graphic algebra; solid geometry. By Dr. F. R. MORRIS: Integral calculus. By Professor W. E. MILNE: Plane analytical geometry. Each class, except the one in "study and research," meets five times a week.

Cornell University, July 3–August 13. Elementary courses in solid geometry, advanced algebra, trigonometry, analytic geometry, and calculus will be given. The following advanced courses are offered: By Professor W. B. CARVER: Teachers' course, selected topics in algebra and geometry. By Professor D. C. GILLESPIE: Projective geometry. By Professor C. F. CRAIG: Analysis. In addition, opportunity for directed reading and research is offered in various fields by Professors F. R. SHARPE, W. B. CARVER, D. C. GILLESPIE, W. A. HURWITZ, F. W. OWENS, and C. F. CRAIG.

Johns Hopkins University, July 6–August 13: By Dr. TERESA COHEN, Analytic geometry, and a choice of work from (*a*) Geometrical transformations, (*b*) Elliptic functions, (*c*) Finite groups.

University of Maine, June 28–August 6. By Professors J. N. HART, H. R. WILLARD, and M. O. TRIPP: Teachers' course in algebra, Teachers' course in geometry, Solid geometry, Plane Trigonometry, College algebra, Analytic geometry, Differential and integral calculus, Advanced analytic geometry, Advanced calculus, Theory of functions.

Massachusetts Institute of Technology, June 21–July 26. By Professor F. S. WOODS: Analytic geometry. By Professors H. B. PHILLIPS and F. L. HITCHCOCK, and Mr. R. D. DOUGLASS: Elementary calculus, and Theoretical mechanics (introductory course). By C. L. E. MOORE: Theoretical mechanics (introductory course). By L. H. RICE: Elements of Differential equations, and Elementary calculus. By S. D. ZELDIN, Elementary calculus.

University of Texas, First Term, June 8–July 20: By Professor E. R. HEDRICK (University of Missouri), Advanced calculus, analytic geometry (straight line and circle). By Professor W. H. ROEVER (Washington University), Descriptive geometry, trigonometry. By Professor H. Y. BENEDICT, Differential calculus, analytic geometry (conic sections). By Professor H. J. ETTLINGER, Fundamentals in elementary mathematics, trigonometry. By Dr. P. M. BATCHELDER, Modern algebra, elementary algebra. By Dr. GOLDIE P. HORTON, Solid analytic geometry, elementary algebra. By Professor E. L. DODD, Least squares, life insurance. Second Term, July 20–August 31: By Professor P. R. RIDER (Washington University), Differential calculus (second part), analytic geometry (conic sections). By Professor C. D. RICE, Advanced mechanics, integral calculus. By Miss MARY DECHERD, Modern algebra (second part), analytic geometry (straight line and circle). By Mr. J. E. BURNAM, Trigonometry, solid geometry.

THE APRIL MEETING OF THE KANSAS SECTION.

The sixth regular meeting of the Kansas Section, postponed from 1918 on account of war conditions, was held at the Kansas State Agricultural College, Manhattan, Kansas, Saturday, April 3, 1920. The chairman, Professor W. A. Harshbarger, presided at both the morning and afternoon sessions. In the absence of the secretary Professor E. B. Stouffer acted as temporary secretary.

The attendance was nineteen, including the following fifteen members of the Association: C. H. Ashton, Lucy T. Dougherty, Ottilia W. Dueker, W. H. Garrett, W. A. Harshbarger, Emma Hyde, S. Lefschetz, T. Lindquist, U. G. Mitchell, Mary W. Newson, B. L. Remick, E. B. Stouffer, W. T. Stratton, J. J. Wheeler, A. E. White.

The following officers were elected: Chairman, Professor W. H. GARRETT, Baker University; Vice-Chairman, Professor J. A. G. SHIRK, State Manual Training Normal School; Secretary, Professor E. B. STOUFFER, University of Kansas. It was decided to hold the next meeting at Topeka in the autumn in connection with the State Teachers Association. The visitors were entertained at a luncheon which was very kindly provided by the members of the department of Mathematics of the Kansas State Agricultural College.

The following six papers were read, the first three at the morning session and the remaining three at the afternoon session:

(1) "Content of the freshman algebra course," by Professor U. G. MITCHELL;

(2) "The bridge between high school and college mathematics," by Professor MARY W. NEWSON;

(3) "A comparative study of the algebraic preparation of the college freshman," by Professor W. H. GARRETT;

(4) "Geometrical representation of the nature of an essential singular point," by Professor C. H. ASHTON;

(5) "The dual of duality," by Professor E. B. STOUFFER;

(6) "Report on tests on college freshmen," by Professor W. T. STRATTON.

The discussion on the first paper was led by Professors Ottilia W. Dueker and A. E. White. There was also a general discussion on this paper and on the paper given by Professor Garrett. Abstracts of the papers follow below, the numbers corresponding to the numbers in the list of titles:

1. In this paper Professor Mitchell presented a list of topics, with a distribution of the number of class-periods suggested for each, to make up a five hour course for one semester designed for students who present but one year of high school algebra for entrance. With certain modifications and omissions the same could be used for a three hour course for one semester designed for students presenting one and one half years of high school algebra for entrance. The selection of material as presented was based upon a logical development of the subject-matter. It was pointed out, however, that the chief concern was not